HL Paper 1

a. Find the sum of the infinite geometric sequence 27, −9, 3, −1, ... [3]

b. Use mathematical induction to prove that for $n \in \mathbb{Z}^+$, [7]

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

Markscheme

a. $r = -\frac{1}{3}$ ***(A1)***

$S_\infty = \frac{27}{1+\frac{1}{3}}$ ***M1***

$S_\infty = \frac{81}{4}$ $(= 20.25)$ ***A1*** ***N1***

[3 marks]

b. Attempting to show that the result is true for $n = 1$ ***M1***

LHS = a **and** RHS $= \frac{a(1-r)}{1-r} = a$ ***A1***

Hence the result is true for $n = 1$

Assume it is true for $n = k$

$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-rk)}{1-r}$ ***M1***

Consider $n = k + 1$:

$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$ ***M1***

$= \frac{a(1-r^k)+ar^k(1-r)}{1-r}$

$= \frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$ ***A1***

Note: Award ***A1*** for an equivalent correct intermediate step.

$= \frac{a-ar^{k+1}}{1-r}$

$= \frac{a(1-r^{k+1})}{1-r}$ ***A1***

Note: Illogical attempted proofs that use the result to be proved would gain ***M1A0A0*** for the last three above marks.

The result is true for $n = k \Rightarrow$ it is true for $n = k + 1$ **and** as it is true for $n = 1$, the result is proved by mathematical induction. ***R1*** ***N0***

Note: To obtain the final ***R1*** mark a reasonable attempt must have been made to prove the $k + 1$ step.

[7 marks]

Examiners report

a. Part (a) was correctly answered by the majority of candidates, although a few found $r = –3$.

b. Part (b) was often started off well, but a number of candidates failed to initiate the $n = k + 1$ step in a satisfactory way. A number of candidates omitted the ‘P(1) is true’ part of the concluding statement.

Find integer values of m and n for which

$$m - n\log_3 2 = 10\log_9 6$$

Markscheme

METHOD 1

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m = \log_3 \left(6^5 2^n\right)$ ***(M1)***

$3^m 2^{-n} = 6^5 = 3^5 \times 2^5$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

Note: First ***M1*** is for any correct change of base, second ***M1*** for writing as a single logarithm, third ***M1*** is for writing 6 as 2×3.

METHOD 2

$m - n\log_3 2 = 10\log_9 6$

$m - n\log_3 2 = 5\log_3 6$ ***M1***

$m - n\log_3 2 = 5\log_3 3 + 5\log_3 2$ ***(M1)***

$m - n\log_3 2 = 5 + 5\log_3 2$ ***(M1)***

$m = 5,\ n = -5$ ***A1***

Note: First ***M1*** is for any correct change of base, second ***M1*** for writing 6 as 2×3 and third ***M1*** is for forming an expression without $\log_3 3$.

[4 marks]

Examiners report

The first stage on this question was to change base, so each logarithm was written in the same base. Some candidates chose to move to base 10 or base e, rather than the more obvious base 3, but a few still successfully reached the correct answer having done this. A large majority though did not seem to know how to change the base of a logarithm.

Simplifying the expression further was a struggle for many candidates.

a. Expand $(x + h)^3$. [2]

b. Hence find the derivative of $f(x) = x^3$ from first principles. [3]

Markscheme

a. $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ ***(M1)A1***

[2 marks]

b. $f'(x) = \lim\limits_{h\to 0} \frac{(x+h)^3 - x^3}{h}$ ***(M1)***

$= \lim\limits_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim\limits_{h\to 0}(3x^2 + 3xh + h^2)$ ***A1***

$= 3x^2$ ***A1***

Note: Do not award final A1 on FT if $= 3x^2$ is not obtained

Note: Final A1 can only be obtained if previous A1 is given

[3 marks]

Total [5 marks]

Examiners report

a. Well done although some did not use the binomial expansion.

b. Fine by those who knew what first principles meant, not by the others.

The fifth term of an arithmetic sequence is equal to 6 and the sum of the first 12 terms is 45.

Find the first term and the common difference.

Markscheme

use of either $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ ***M1***

$u_1 + 4d = 6$ ***(A1)***

$\frac{12}{2}(2u_1 + 11d) = 45$ ***(A1)***

$\Rightarrow 4u_1 + 22d = 15$

attempt to solve simultaneous equations ***M1***

$4(6-4d) + 22d = 15$

$6d = -9 \Rightarrow d = -1.5$ ***A1***

$u_1 = 12$ ***A1***

[6 marks]

Examiners report

Most candidates tackled this question through the use of the standard formula for arithmetic series. Others attempted a variety of trial and improvement approaches with varying degrees of success.

The complex numbers $z_1 = 2 - 2\mathrm{i}$ and $z_2 = 1 - \sqrt{3}\mathrm{i}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

a. Find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where a , $b \in \mathbb{Z}^+$. [3]

b. Calculate $\mathrm{A\hat{O}B}$ in terms of π. [3]

Markscheme

a. $\mathrm{AB} = \sqrt{1^2 + (2 - \sqrt{3})^2}$ ***M1***

$= \sqrt{8 - 4\sqrt{3}}$ ***A1***

$= 2\sqrt{2 - \sqrt{3}}$ ***A1***

[3 marks]

b. **METHOD 1**

$\arg z_1 = -\frac{\pi}{4}$ $\arg z_2 = -\frac{\pi}{3}$ ***A1A1***

Note: Allow $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Note: Allow degrees at this stage.

$\mathrm{A\hat{O}B} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ (accept $-\frac{\pi}{12}$) ***A1***

Note: Allow ***FT*** for final ***A1***.

METHOD 2

attempt to use scalar product or cosine rule ***M1***

$\cos \mathrm{A\hat{O}B} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ ***A1***

$\mathrm{A\hat{O}B} = \frac{\pi}{12}$ ***A1***

[3 marks]

Examiners report

a. It was disappointing to note the lack of diagram in many solutions. Most importantly the lack of understanding of the notation AB was apparent. Teachers need to make sure that students are aware of correct notation as given in the outline. A number used the cosine rule but then confused the required angle or sides.

b. It was disappointing to note the lack of diagram in many solutions. Most importantly the lack of understanding of the notation AB was apparent. Teachers need to make sure that students are aware of correct notation as given in the outline. A number used the cosine rule but then confused the required angle or sides.

Let $z = 1 - \cos 2\theta - \mathrm{i}\sin 2\theta,\ z \in \mathbb{C},\ 0 \leqslant \theta \leqslant \pi$.

a. Solve $2\sin(x + 60^\circ) = \cos(x + 30^\circ),\ 0^\circ \leqslant x \leqslant 180^\circ$. [5]

b. Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$. [3]

c.i. Find the modulus and argument of z in terms of θ. Express each answer in its simplest form. [9]

c.ii. Hence find the cube roots of z in modulus-argument form. [5]

Markscheme

a. $2\sin(x + 60^\circ) = \cos(x + 30^\circ)$

$2(\sin x \cos 60^\circ + \cos x \sin 60^\circ) = \cos x \cos 30^\circ - \sin x \sin 30^\circ$ ***(M1)(A1)***

$2\sin x \times \frac{1}{2} + 2\cos x \times \frac{\sqrt{3}}{2} = \cos x \times \frac{\sqrt{3}}{2} - \sin x \times \frac{1}{2}$ ***A1***

$\Rightarrow \frac{3}{2}\sin x = -\frac{\sqrt{3}}{2}\cos x$

$\Rightarrow \tan x = -\frac{1}{\sqrt{3}}$ ***M1***

$\Rightarrow x = 150^\circ$ ***A1***

[5 marks]

b. **EITHER**

choosing two appropriate angles, for example 60° and 45° ***M1***

$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$ and

$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$ ***(A1)***

$\sin 105^\circ + \cos 105^\circ = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$ ***A1***

$= \frac{1}{\sqrt{2}}$ ***AG***

OR

attempt to square the expression ***M1***

$(\sin 105^\circ + \cos 105^\circ)^2 = \sin^2 105^\circ + 2\sin 105^\circ \cos 105^\circ + \cos^2 105^\circ$

$(\sin 105^\circ + \cos 105^\circ)^2 = 1 + \sin 210^\circ$ ***A1***

$= \frac{1}{2}$ ***A1***

$\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$ ***AG***

[3 marks]

c.i. **EITHER**

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$|z| = \sqrt{(1 - \cos 2\theta)^2 + (\sin 2\theta)^2}$ ***M1***

$|z| = \sqrt{1 - 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$ ***A1***

$= \sqrt{2}\sqrt{(1 - \cos 2\theta)}$ ***A1***

$= \sqrt{2(2\sin^2\theta)}$

$= 2\sin\theta$ ***A1***

let $\arg(z) = \alpha$

$\tan\alpha = -\frac{\sin 2\theta}{1 - \cos 2\theta}$ ***M1***

$= \frac{-2\sin\theta\cos\theta}{2\sin^2\theta}$ ***(A1)***

$= -\cot\theta$ ***A1***

$\arg(z) = \alpha = -\arctan\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)$ ***A1***

$= \theta - \frac{\pi}{2}$ ***A1***

OR

$z = (1 - \cos 2\theta) - \mathrm{i}\sin 2\theta$

$= 2\sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$ ***M1A1***

$= 2\sin\theta(\sin\theta - \mathrm{i}\cos\theta)$ ***(A1)***

$= -2\mathrm{i}\sin\theta(\cos\theta + \mathrm{i}\sin\theta)$ ***M1A1***

$= 2\sin\theta\left(\cos\left(\theta - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \frac{\pi}{2}\right)\right)$ ***M1A1***

$|z| = 2\sin\theta$ ***A1***

$\arg(z) = \theta - \frac{\pi}{2}$ ***A1***

[9 marks]

c.ii. attempt to apply De Moivre's theorem ***M1***

$$(1 - \cos 2\theta - \mathrm{i}\sin 2\theta)^{\frac{1}{3}} = 2^{\frac{1}{3}}(\sin\theta)^{\frac{1}{3}}\left[\cos\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\theta - \frac{\pi}{2} + 2n\pi}{3}\right)\right]$$ ***A1A1A1***

Note: ***A1*** for modulus, ***A1*** for dividing argument of z by 3 and ***A1*** for $2n\pi$.

Hence cube roots are the above expression when $n = -1,\ 0,\ 1$. Equivalent forms are acceptable. ***A1***

[5 marks]

Examiners report

a. [N/A]
[N/A]

b.i. [N/A]
c.ii. [N/A]

Consider the distinct complex numbers $z = a + \mathrm{i}b,\ w = c + \mathrm{i}d$, where $a,\ b,\ c,\ d \in \mathbb{R}$.

a. Find the real part of $\frac{z+w}{z-w}$. [4]

b. Find the value of the real part of $\frac{z+w}{z-w}$ when $|z| = |w|$. [2]

Markscheme

a. $\frac{z+w}{z-w} = \frac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)}$

$= \frac{(a+c)+\mathrm{i}(b+d)}{(a-c)+\mathrm{i}(b-d)} \times \frac{(a-c)-\mathrm{i}(b-d)}{(a-c)-\mathrm{i}(b-d)}$ ***M1A1***

real part $= \frac{(a+c)(a-c)+(b+d)(b-d)}{(a-c)^2+(b-d)^2} = \left(\frac{a^2-c^2+b^2-d^2}{(a-c)^2+(b-d)^2}\right)$ ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

[4 marks]

b. $|z| = |w| \Rightarrow a^2 + b^2 = c^2 + d^2$ ***R1***

hence real part = 0 ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the complex number $\omega = \frac{z+\mathrm{i}}{z+2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r\,\mathrm{cis}\,\theta$.

(b) Prove that $\omega = \frac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$.

(c) **Hence** show that when $\mathrm{Re}(\omega) = 1$ the points $(x,\ y)$ lie on a straight line, l_1, and write down its gradient.

(d) Given $\arg(z) = \arg(\omega) = \frac{\pi}{4}$, find $|z|$.

Markscheme

(a) **METHOD 1**

$\frac{z+\mathrm{i}}{z+2} = \mathrm{i}$

$z + \mathrm{i} = \mathrm{i}z + 2\mathrm{i}$ ***M1***

$(1-\mathrm{i})z=\mathrm{i}$ ***A1***

$z=\frac{\mathrm{i}}{1-\mathrm{i}}$ ***A1***

EITHER

$z=\frac{\operatorname{cis}\left(\frac{\pi}{2}\right)}{\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}$ ***M1***

$z=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)\ \left(\text{or }\frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{4\pi}{4}\right)\right)$ ***A1A1***

OR

$z=\frac{-1+\mathrm{i}}{2}\ \left(=-\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)$ ***M1***

$z=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)\ \left(\text{or }\frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)\right)$ ***A1A1***

[6 marks]

METHOD 2

$\mathrm{i}=\frac{x+\mathrm{i}(y+1)}{x+2+\mathrm{i}y}$ ***M1***

$x+\mathrm{i}(y+1)=-y+\mathrm{i}(x+2)$ ***A1***

$x=-y;\ x+2=y+1$ ***A1***

solving, $x=-\frac{1}{2};\ y=\frac{1}{2}$ ***A1***

$z=-\frac{1}{2}+\frac{1}{2}\mathrm{i}$

$z=\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)\ \left(\text{or }\frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)\right)$ ***A1A1***

Note: Award ***A1*** fort the correct modulus and ***A1*** for the correct argument, but the final answer must be in the form r cis θ. Accept 135° for the argument.

[6 marks]

(b) substituting $z=x+\mathrm{i}y$ to obtain $w=\frac{x+(y+1)\mathrm{i}}{(x+2)+y\mathrm{i}}$ ***(A1)***

use of $(x+2)-y\mathrm{i}$ to rationalize the denominator ***M1***

$\omega=\frac{x(x+2)+y(y+1)+\mathrm{i}(-xy+(y+1)(x+2))}{(x+2)^2+y^2}$ ***A1***

$=\frac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$ ***AG***

[3 marks]

(c) $\operatorname{Re}\omega=\frac{x^2+2x+y^2+y}{(x+2)^2+y^2}=1$ ***M1***

$\Rightarrow x^2+2x+y^2+y=x^2+4x+4+y^2$ ***A1***

$\Rightarrow y=2x+4$ ***A1***

which has gradient $m=2$ ***A1***

[4 marks]

(d) **EITHER**

$\arg(z)=\frac{\pi}{4}\Rightarrow x=y$ (and $x,\ y>0$) ***(A1)***

$\omega=\frac{2x^2+3x}{(x+2)^2+x^2}+\frac{\mathrm{i}(3x+2)}{(x+2)^2+x^2}$

if $\arg(\omega) = \theta \Rightarrow \tan\theta = \frac{3x+2}{2x^2+3x}$ ***(M1)***

$\frac{3x+2}{2x^2+3x} = 1$ ***M1A1***

OR

$\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ (and $x,\ y > 0$) ***A1***

$\arg(w) = \frac{\pi}{4} \Rightarrow x^2 + 2x + y^2 + y = x + 2y + 2$ ***M1***

solve simultaneously ***M1***

$x^2 + 2x + x^2 + x = x + 2x + 2$ (or equivalent) ***A1***

THEN

$x^2 = 1$

$x = 1$ (as $x > 0$) ***A1***

Note: Award ***A0*** for $x = \pm 1$.

$|z| = \sqrt{2}$ ***A1***

Note: Allow ***FT*** from incorrect values of x.

[6 marks]

Total [19 marks]

Examiners report

Many candidates knew what had to be done in (a) but algebraic errors were fairly common. Parts (b) and (c) were well answered in general. Part (d), however, proved beyond many candidates who had no idea how to convert the given information into mathematical equations.

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a. Find three distinct roots of the equation $8z^3 + 27 = 0,\ z \in \mathbb{C}$ giving your answers in modulus-argument form. [6]

b. The roots are represented by the vertices of a triangle in an Argand diagram. [3]

Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$.

Markscheme

a. **METHOD 1**

$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$ ***M1(A1)***

$= \frac{27}{8}(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi))$ ***(A1)***

$z = \frac{3}{2}\left(\cos\left(\frac{\pi+2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi+2n\pi}{3}\right)\right)$ ***M1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$. ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{\mathrm{i}\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z+3)$ is a factor

Attempt to use long division or factor theorem: ***M1***

$\Rightarrow 8z^3 + 27 = (2z+3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ ***A1***

Attempt to solve quadratic: ***M1***

$z = \frac{3 \pm 3\sqrt{3}\mathrm{i}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$. ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{\mathrm{i}\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

METHOD 3

$8z^3 + 27 = 0$

Substitute $z = x + \mathrm{i}y$ ***M1***

$8(x^3 + 3\mathrm{i}x^2y - 3xy^2 - \mathrm{i}y^3) + 27 = 0$

$\Rightarrow 8x^3 - 24xy^2 + 27 = 0$ and $24x^2y - 8y^3 = 0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y(3x^2 - y^2) = 0$

$y = 0,\ y = x\sqrt{3},\ y = -x\sqrt{3}$

$\Rightarrow \left(x = -\frac{3}{2},\ y = 0\right),\ x = \frac{3}{4},\ y = \pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi),$

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{\mathrm{i}\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

b. **EITHER**

Valid attempt to use $\text{area} = 3\left(\frac{1}{2}ab\sin C\right)$ ***M1***

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$ ***A1A1***

Note: Award ***A1*** for correct sides, ***A1*** for correct sin C.

OR

Valid attempt to use $\text{area} = \frac{1}{2}\text{base} \times \text{height}$ ***M1***

$\text{area} = \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2}\right) \times \frac{6\sqrt{3}}{4}$ ***A1A1***

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$ ***AG***

[3 marks]

Total [9 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the complex numbers $z = 1 + 2\mathrm{i}$ and $w = 2 + a\mathrm{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$; ;

(b) $\mathrm{Re}(zw) = 2\,\mathrm{Im}(zw)$.

Markscheme

(a) $|z| = \sqrt{5}$ and $|w| = \sqrt{4 + a^2}$

$|w| = 2|z|$

$\sqrt{4 + a^2} = 2\sqrt{5}$

attempt to solve equation ***M1***

Note: Award ***M0*** if modulus is not used.

$a = \pm 4$ ***A1A1*** ***N0***

(b) $zw = (2 - 2a) + (4 + a)\mathrm{i}$ ***A1***

forming equation $2 - 2a = 2(4 + a)$ ***M1***

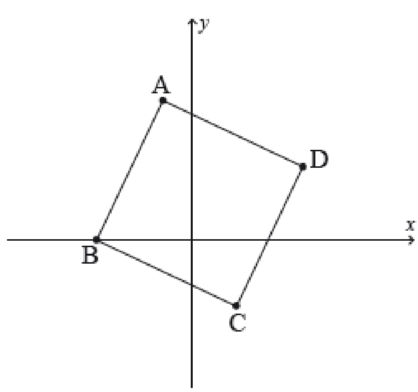
$a = -\frac{3}{2}$ ***A1*** ***N0***

[6 marks]

Examiners report

Most candidates made good attempts to answer this question. Weaker candidates did not get full marks due to difficulties recognizing the notation and working with modulus of a complex number.

In the following Argand diagram the point A represents the complex number $-1 + 4\mathrm{i}$ and the point B represents the complex number $-3 + 0\mathrm{i}$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



Markscheme

C represents the complex number $1 - 2\mathrm{i}$ ***A2***

D represents the complex number $3+2\mathrm{i}$ ***A2***

[4 marks]

Examiners report

[N/A]

Use mathematical induction to prove that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$.

Markscheme

let $\mathrm{P}(n)$ be the proposition that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$

consider P(1):

when $n=1,\ n(n^2+5)=1\times(1^2+5)=6$ and so P(1) is true ***R1***

assume $\mathrm{P}(k)$ is true *ie*, $k(k^2+5)=6m$ where $k,\ m \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$".

consider $\mathrm{P}(k+1)$:

$(k+1)\left((k+1)^2+5\right)$ ***M1***

$=(k+1)(k^2+2k+6)$

$=k^3+3k^2+8k+6$ ***(A1)***

$=(k^3+5k)+(3k^2+3k+6)$ ***A1***

$=k(k^2+5)+3k(k+1)+6$ ***A1***

$k(k+1)$ is even hence all three terms are divisible by 6 ***R1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and P(1) is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[8 marks]

Examiners report

This proved to be a good discriminator. The average candidate seemed able to work towards $\mathrm{P}(k+1)=k^3+3k^2+8k+6$, and a number made some further progress.

Unfortunately, even otherwise good candidates are still writing down incorrect or incomplete induction statements, such as 'Let $n=k$' rather than 'Suppose true for $n=k$' (or equivalent).

It was also noted than an increasing number of candidates this session assumed '$\mathrm{P}(n)$ to be true' before going to consider $\mathrm{P}(n+1)$. Showing a lack of understanding of the induction argument, these approaches scored very few marks.

It is given that $\log_2 y+\log_4 x+\log_4 2x=0$.

a. Show that $\log_{r^2} x = \frac{1}{2}\log_r x$ where $r,\ x \in \mathbb{R}^+$. [2]

b. Express y in terms of x. Give your answer in the form $y = px^q$, where *p* , *q* are constants. [5]

c. The region *R*, is bounded by the graph of the function found in part (b), the *x*-axis, and the lines $x = 1$ and $x = \alpha$ where $\alpha > 1$. The area of *R* is $\sqrt{2}$. [5]

Find the value of α.

Markscheme

a. **METHOD 1**

$\log_{r^2} x = \frac{\log_r x}{\log_r r^2} \left(= \frac{\log_r x}{2\log_r r}\right)$ ***M1A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

METHOD 2

$\log_{r^2} x = \frac{1}{\log_x r^2}$ ***M1***

$= \frac{1}{2\log_x r}$ ***A1***

$= \frac{\log_r x}{2}$ ***AG***

[2 marks]

b. **METHOD 1**

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \log_4 2x^2 = 0$ ***M1***

$\log_2 y + \frac{1}{2}\log_2 2x^2 = 0$ ***M1***

$\log_2 y = -\frac{1}{2}\log_2 2x^2$

$\log_2 y = \log_2 \left(\frac{1}{\sqrt{2x}}\right)$ ***M1A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

METHOD 2

$\log_2 y + \log_4 x + \log_4 2x = 0$

$\log_2 y + \frac{1}{2}\log_2 x + \frac{1}{2}\log_2 2x = 0$ ***M1***

$\log_2 y + \log_2 x^{\frac{1}{2}} + \log_2 (2x)^{\frac{1}{2}} = 0$ ***M1***

$\log_2 \left(\sqrt{2}xy\right) = 0$ ***M1***

$\sqrt{2}xy = 1$ ***A1***

$y = \frac{1}{\sqrt{2}}x^{-1}$ ***A1***

Note: For the final ***A*** mark, y must be expressed in the form px^q.

[5 marks]

c. the area of *R* is $\int_1^{\alpha} \frac{1}{\sqrt{2}}x^{-1}\mathrm{d}x$ ***M1***

$= \left[\frac{1}{\sqrt{2}}\ln x\right]_1^{\alpha}$ ***A1***

$= \frac{1}{\sqrt{2}}\ln \alpha$ ***A1***

$\frac{1}{\sqrt{2}}\ln \alpha = \sqrt{2}$ ***M1***

$\alpha = \mathrm{e}^2$ ***A1***

Note: Only follow through from part (b) if y is in the form $y = px^q$

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

An 81 metre rope is cut into *n* pieces of increasing lengths that form an arithmetic sequence with a common difference of *d* metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of *n* and *d* .

Markscheme

$81 = \frac{n}{2}(1.5 + 7.5)$ ***M1***

$\Rightarrow n = 18$ ***A1***

$1.5 + 17d = 7.5$ ***M1***

$\Rightarrow d = \frac{6}{17}$ ***A1*** ***N0***

[4 marks]

Examiners report

There were many totally correct solutions to this question, but a number of candidates found two simultaneous equations and then spent a lot of time and working trying, often unsuccessfully, to solve these equations.

Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

Markscheme

METHOD 1

$z = (2 - \mathrm{i})(z + 2)$ ***M1***

$= 2z + 4 - \mathrm{i}z - 2\mathrm{i}$

$z(1 - \mathrm{i}) = -4 + 2\mathrm{i}$

$z = \frac{-4+2\mathrm{i}}{1-\mathrm{i}}$ ***A1***

$z = \frac{-4+2\mathrm{i}}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}}$ ***M1***

$= -3 - \mathrm{i}$ ***A1***

METHOD 2

let $z = a + \mathrm{i}b$
$\frac{a+\mathrm{i}b}{a+\mathrm{i}b+2} = 2 - \mathrm{i}$ ***M1***
$a + \mathrm{i}b = (2 - i)((a + 2) + \mathrm{i}b)$
$a + \mathrm{i}b = 2(a + 2) + 2b\mathrm{i} - \mathrm{i}(a + 2) + b$
$a + \mathrm{i}b = 2a + b + 4 + (2b - a - 2)\mathrm{i}$
attempt to equate real and imaginary parts ***M1***
$a = 2a + b + 4 (\Rightarrow a + b + 4 = 0)$
and $b = 2b - a - 2 (\Rightarrow -a + b - 2 = 0)$ ***A1***

Note: Award ***A1*** for two correct equations.

$b = -1; a = -3$ ***A1***

$z = -3 - \mathrm{i}$

[4 marks]

Examiners report

A number of different methods were adopted in this question with some candidates working through their method to a correct answer. However many other candidates either stopped with z still expressed as a quotient of two complex numbers or made algebraic mistakes.

Given that $(4 - 5\mathrm{i})m + 4n = 16 + 15\mathrm{i}$, where $\mathrm{i}^2 = -1$, find *m* and *n* if

a. *m* and *n* are real numbers; [3]

b. *m* and *n* are conjugate complex numbers. [4]

Markscheme

a. attempt to equate real and imaginary parts ***M1***

equate real parts: $4m + 4n = 16$; equate imaginary parts: $-5m = 15$ ***A1***
$\Rightarrow m = -3,\ n = 7$ ***A1***
[3 marks]

b. let $m = x + \mathrm{i}y,\ n = x - \mathrm{i}y$ ***M1***

$\Rightarrow (4 - 5\mathrm{i})(x + \mathrm{i}y) + 4(x - \mathrm{i}y) = 16 + 15\mathrm{i}$
$\Rightarrow 4x - 5\mathrm{i}x + 4\mathrm{i}y + 5y + 4x - 4\mathrm{i}y = 16 + 15\mathrm{i}$
attempt to equate real and imaginary parts ***M1***
$8x + 5y = 16,\ -5x = 15$ ***A1***
$\Rightarrow x = -3,\ y = 8$ ***A1***
$(\Rightarrow m = -3 + 8\mathrm{i},\ n = -3 - 8\mathrm{i})$
[4 marks]

Examiners report

a. Part (a) was generally well answered. In (b), however, some candidates put $m = a + \mathrm{i}b$ and $n = c + \mathrm{i}d$ which gave four equations for two unknowns so that no further progress could be made.

b. Part (a) was generally well answered. In (b), however, some candidates put $m = a + \mathrm{i}b$ and $n = c + \mathrm{i}d$ which gave four equations for two unknowns so that no further progress could be made.

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

The region S is bounded by the curve $y = \sin x \cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

a. Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta,\ n \in \mathbb{Z}^+$. [2]

b. Expand $\left(z + z^{-1}\right)^4$. [1]

c. Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where $p,\ q$ and r are constants to be determined. [4]

d. Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. [3]

e. Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \mathrm{d}\theta$. [3]

f. S is rotated through 2π radians about the x-axis. Find the value of the volume generated. [4]

g. (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$. [3]

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta$ in terms of k.

Markscheme

a. $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***
$= 2\cos n\theta$ ***AG***
[2 marks]

b. (b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

Note: Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

c. **METHOD 1**

$\left(z+z^{-1}\right)^4=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$ ***M1***

$(2\cos\theta)^4=2\cos 4\theta+8\cos 2\theta+6$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$ ***A1***

$\left(\text{or } p=\frac{1}{8},\ q=\frac{1}{2},\ r=\frac{3}{8}\right)$

METHOD 2

$\cos^4\theta=\left(\frac{\cos 2\theta+1}{2}\right)^2$ ***M1***

$=\frac{1}{4}(\cos^2 2\theta+2\cos 2\theta+1)$ ***A1***

$=\frac{1}{4}\left(\frac{\cos 4\theta+1}{2}+2\cos 2\theta+1\right)$ ***A1***

$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$ ***A1***

$\left(\text{or } p=\frac{1}{8},\ q=\frac{1}{2},\ r=\frac{3}{8}\right)$

[4 marks]

d. $\left(z+z^{-1}\right)^6=z^6+6z^5\left(\frac{1}{z}\right)+15z^4\left(\frac{1}{z^2}\right)+20z^3\left(\frac{1}{z^3}\right)+15z^2\left(\frac{1}{z^4}\right)+6z\left(\frac{1}{z^5}\right)+\frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6=\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$

$(2\cos\theta)^6=2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta=\frac{1}{32}\cos 6\theta+\frac{3}{16}\cos 4\theta+\frac{15}{32}\cos 2\theta+\frac{5}{16}$ ***AG***

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

e. $\int_0^{\frac{\pi}{2}}\cos^6\theta\,\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\left(\frac{1}{32}\cos 6\theta+\frac{3}{16}\cos 4\theta+\frac{15}{32}\cos 2\theta+\frac{5}{16}\right)\mathrm{d}\theta$

$=\left[\frac{1}{192}\sin 6\theta+\frac{3}{64}\sin 4\theta+\frac{15}{64}\sin 2\theta+\frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$=\frac{5\pi}{32}$ ***A1***

[3 marks]

f. $\mathrm{V}=\pi\int_0^{\frac{\pi}{2}}\sin^2 x\cos^4 x\,\mathrm{d}x$ ***M1***

$=\pi\int_0^{\frac{\pi}{2}}\cos^4 x\,\mathrm{d}x-\pi\int_0^{\frac{\pi}{2}}\cos^6 x\,\mathrm{d}x$ ***M1***

$\int_0^{\frac{\pi}{2}}\cos^4 x\,\mathrm{d}x=\frac{3\pi}{16}$ ***A1***

$\mathrm{V}=\frac{3\pi^2}{16}-\frac{5\pi^2}{32}=\frac{\pi^2}{32}$ ***A1***

Note: Follow through from an incorrect *r* in (c) provided the final answer is positive.

g. (i) constant term = $\binom{2k}{k}=\frac{(2k)!}{k!k!}=\frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}}\cos^{2k}\theta\,\mathrm{d}\theta=\frac{(2k)!\pi}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}}\cos^{2k}\theta\,\mathrm{d}\theta=\frac{(2k)!\pi}{2^{2k+1}(k!)^2}\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Examiners report

a. Part a) has appeared several times before, though with it again being a ‘show that’ question, some candidates still need to be more aware of the need to show every step in their working, including the result that $\sin(-n\theta) = -\sin(n\theta)$.

b. Part b) was usually answered correctly.

c. Part c) was again often answered correctly, though some candidates often less successfully utilised a trig-only approach rather than taking note of part b).

d. Part d) was a good source of marks for those who kept with the spirit of using complex numbers for this type of question. Some limited attempts at trig-only solutions were seen, and correct solutions using this approach were extremely rare.

e. Part e) was well answered, though numerical slips were often common. A small number integrated $\sin n\theta$ as $n \cos n\theta$.

A large number of candidates did not realise the help that part e) inevitably provided for part f). Some correctly expressed the volume as $\pi \int \cos^4 x \mathrm{d}x - \pi \int \cos^6 x \mathrm{d}x$ and thus gained the first 2 marks but were able to progress no further. Only a small number of able candidates were able to obtain the correct answer of $\frac{\pi^2}{32}$.

f. [N/A]

g. Part g) proved to be a challenge for the vast majority, though it was pleasing to see some of the highest scoring candidates gain all 3 marks.

The geometric sequence u_1, u_2, u_3, … has common ratio *r*.

Consider the sequence $A = \left\{a_n = \log_2 |u_n| : n \in \mathbb{Z}^+\right\}$.

a. Show that *A* is an arithmetic sequence, stating its common difference *d* in terms of *r*. [4]

b. A particular geometric sequence has $u_1 = 3$ and a sum to infinity of 4. [3]

Find the value of *d*.

Markscheme

a. **METHOD 1**

state that $u_n = u_1 r^{n-1}$ (or equivalent) ***A1***

attempt to consider a_n and use of at least one log rule ***M1***

$\log_2 |u_n| = \log_2 |u_1| + (n-1)\log_2 |r|$ ***A1***

(which is an AP) with $d = \log_2 |r|$ (and 1st term $\log_2 |u_1|$) ***A1***

so A is an arithmetic sequence ***AG***

Note: Condone absence of modulus signs.

Note: The final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

[4 marks]

METHOD 2

consideration of $(d =)\, a_{n+1} - a_n$ ***M1***

$(d) = \log_2 |u_{n+1}| - \log_2 |u_n|$

$(d) = \log_2 \left|\frac{u_{n+1}}{u_n}\right|$ ***M1***

$(d) = \log_2 |r|$ ***A1***

which is constant ***R1***

Note: Condone absence of modulus signs.

Note: The final ***A*** mark may be awarded independently.

Note: Consideration of the first two or three terms only will score ***M0***.

b. attempting to solve $\frac{3}{1-r} = 4$ ***M1***

$r = \frac{1}{4}$ ***A1***

$d = -2$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the following equations, where $a\,,\, b \in \mathbb{R}$:

$x + 3y + (a-1)z = 1$
$2x + 2y + (a-2)z = 1$
$3x + y + (a-3)z = b.$

a. If each of these equations defines a plane, show that, for any value of a , the planes do not intersect at a unique point. [3]

b. Find the value of b for which the intersection of the planes is a straight line. [4]

Markscheme

a. **METHOD 1**

$\det\begin{pmatrix} 1 & 3 & a-1 \\ 2 & 2 & a-2 \\ 3 & 1 & a-3 \end{pmatrix}$ ***M1***

$= 1\,(2(a-3) - (a-2)) - 3\,(2(a-3) - 3(a-2)) + (a-1)(2-6)$

(or equivalent) ***A1***

$= 0$ (therefore there is no unique solution) ***A1***

[3 marks]

METHOD 2

$\left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 2 & 2 & a-2 & 1 \\ 3 & 1 & a-3 & b \end{array}\right) : \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & -8 & -2a & b-3 \end{array}\right)$ ***M1A1***

$: \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$ (and 3 zeros imply no unique solution) ***A1***

[3 marks]

b. **METHOD 1**

$$\left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 2 & 2 & a-2 & 1 \\ 3 & 1 & a-3 & b \end{array}\right) : \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & -8 & -2a & b-3 \end{array}\right) \quad \textbf{\textit{M1A1}}$$

$$: \left(\begin{array}{ccc|c} 1 & 3 & a-1 & 1 \\ 0 & -4 & -a & -1 \\ 0 & 0 & 0 & b-1 \end{array}\right) \quad \textbf{\textit{A1}}$$

$b = 1$ ***A1*** ***N2***

Note: Award ***M1*** for an attempt to use row operations.

[4 marks]

METHOD 2

$b = 1$ ***A4***

Note: Award ***A4*** only if " $b - 1$ " seen in (a).

[4 marks]

Examiners report

a. The best candidates used row reduction correctly in part a) and were hence able to deduce $b = 1$ in part b) for an easy final 4 marks. The determinant method was often usefully employed in part a).

b. The best candidates used row reduction correctly in part a) and were hence able to deduce $b = 1$ in part b) for an easy final 4 marks. The determinant method was often usefully employed in part a).

Consider the complex numbers $z_1 = 1 + \sqrt{3}\mathrm{i}$, $z_2 = 1 + \mathrm{i}$ and $w = \frac{z_1}{z_2}$.

a.i. By expressing z_1 and z_2 in modulus-argument form write down the modulus of w; [3]

a.ii. By expressing z_1 and z_2 in modulus-argument form write down the argument of w. [1]

b. Find the smallest positive integer value of n, such that w^n is a real number. [2]

Markscheme

a.i. $z_1 = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$|w| = \sqrt{2}$ ***A1***

[3 marks]

a.ii. $z_1 = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$\arg w = \frac{\pi}{12}$ ***A1***

Notes: Allow ***FT*** from incorrect answers for z_1 and z_2 in modulus-argument form.

[1 mark]

b. **EITHER**

$\sin\left(\frac{\pi n}{12}\right) = 0$ ***(M1)***

OR

$\arg(w^n) = \pi$ ***(M1)***

$\frac{n\pi}{12} = \pi$

THEN

$\therefore n = 12$ ***A1***

[2 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

Consider the complex numbers
$z_1 = 2\sqrt{3}\mathrm{cis}\frac{3\pi}{2}$ and $z_2 = -1 + \sqrt{3}\mathrm{i}$.

a. (i) Write down z_1 in Cartesian form. [3]

(ii) Hence determine $(z_1 + z_2)^*$ in Cartesian form.

b. (i) Write z_2 in modulus-argument form. [6]

(ii) Hence solve the equation $z^3 = z_2$.

c. Let $z = r\ \mathrm{cis}\theta$, where $r \in \mathbb{R}^+$ and $0 \leqslant \theta < 2\pi$. Find all possible values of *r* and θ , [6]

(i) if $z^2 = (1 + z_2)^2$;
(ii) if $z = -\frac{1}{z_2}$.

d. Find the smallest positive value of *n* for which $\left(\frac{z_1}{z_2}\right)^n \in \mathbb{R}^+$. [4]

Markscheme

a. (i) $z_1 = 2\sqrt{3}\mathrm{cis}\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\mathrm{i}$ ***A1***

(ii) $z_1 + z_2 = -2\sqrt{3}\mathrm{i} - 1 + \sqrt{3}\mathrm{i} = -1 - \sqrt{3}\mathrm{i}$ ***A1***

$(z_1 + z_2)^* = -1 + \sqrt{3}\mathrm{i}$ ***A1***

[3 marks]

b. (i) $|z_2| = 2$
$\tan\theta = -\sqrt{3}$ ***(M1)***
z_2 lies on the second quadrant
$\theta = \arg z_2 = \frac{2\pi}{3}$
$z_2 = 2\mathrm{cis}\frac{2\pi}{3}$ ***A1A1***

(ii) attempt to use De Moivre's theorem ***M1***
$z = \sqrt[3]{2}\,\mathrm{cis}\frac{\frac{2\pi}{3}+2k\pi}{3},\ k = 0, 1 \text{ and } 2$
$z = \sqrt[3]{2}\,\mathrm{cis}\frac{2\pi}{9},\ \sqrt[3]{2}\,\mathrm{cis}\frac{8\pi}{9},\ \sqrt[3]{2}\,\mathrm{cis}\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\mathrm{cis}\left(\frac{-4\pi}{9}\right)\right)$ ***A1A1***
Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z .

[6 marks]

c. (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\mathrm{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\mathrm{i}\right)$ ***M1***
$z = \sqrt{3}\,\mathrm{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\mathrm{cis}\frac{3\pi}{2}\left(= \sqrt{3}\,\mathrm{cis}\left(\frac{-\pi}{2}\right)\right)$ ***A1A1***
so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$
Note: Accept $r\,\mathrm{cis}(\theta)$ form.

METHOD 2
$z^2 = \left(1 - 1 + \sqrt{3}\mathrm{i}\right)^2 = -3 \Rightarrow z^2 = 3\mathrm{cis}\left((2n+1)\pi\right)$ ***M1***
$r^2 = 3 \Rightarrow r = \sqrt{3}$ ***A1***
$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \leqslant \theta < 2\pi$) ***A1***
Note: Accept $r\,\mathrm{cis}(\theta)$ form.

(ii) **METHOD 1**
$z = -\frac{1}{2\mathrm{cis}\frac{2\pi}{3}} \Rightarrow z = \frac{\mathrm{cis}\pi}{2\mathrm{cis}\frac{2\pi}{3}}$ ***M1***
$\Rightarrow z = \frac{1}{2}\mathrm{cis}\frac{\pi}{3}$
so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ ***A1A1***
METHOD 2
$z_1 = -\frac{1}{-1+\sqrt{3}\mathrm{i}} \Rightarrow z_1 = -\frac{-1-\sqrt{3}\mathrm{i}}{\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)}$ ***M1***
$z = \frac{1+\sqrt{3}\mathrm{i}}{4} \Rightarrow z = \frac{1}{2}\mathrm{cis}\frac{\pi}{3}$
so $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ ***A1A1***
[6 marks]

d. $\frac{z_1}{z_2} = \sqrt{3}\,\mathrm{cis}\frac{5\pi}{6}$ ***(A1)***
$\left(\frac{z_1}{z_2}\right)^n = \sqrt{3}^n\mathrm{cis}\frac{5n\pi}{6}$ ***A1***
equating imaginary part to zero and attempting to solve ***M1***
obtain $n = 12$ ***A1***
Note: Working which only includes the argument is valid.

[4 marks]

Examiners report

a. Many candidates were perhaps fortunate in this question due to there being several follow through marks available. Part a) was often done correctly. In part b), incorrect answers of $2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$ were common, though many of these candidates often applied De Moivre's Theorem correctly to their answers. In c) the majority found $z^2 = -3$ but could then get no further. The second part was often poorly done, with those rationalising the Cartesian form of z_1 having the most success. Part d) posed problems for a great many, and correct solutions were rarely seen. $\mathrm{cis}\left(\frac{5n\pi}{6}\right)$ was often seen, but then finding $n = 12$ proved to be a step too far for many. In general, the manipulation of complex numbers in polar form is not well understood.

b. Many candidates were perhaps fortunate in this question due to there being several follow through marks available. Part a) was often done correctly. In part b), incorrect answers of $2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$ were common, though many of these candidates often applied De Moivre's Theorem correctly to their answers. In c) the majority found $z^2 = -3$ but could then get no further. The second part was often poorly done, with those rationalising the Cartesian form of z_1 having the most success. Part d) posed problems for a great many, and correct solutions were rarely seen. $\mathrm{cis}\left(\frac{5n\pi}{6}\right)$ was often seen, but then finding $n = 12$ proved to be a step too far for many. In general, the manipulation of complex numbers in polar form is not well understood.

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. (a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference. [14]

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio.

(c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r+1)2^{r-1}$.

d. Using mathematical induction, prove that [7]

$$\sum_{r=1}^{n}(r+1)2^{r-1} = n2^n,\ n \in \mathbb{Z}^+.$$

Markscheme

. (a) $S_6 = 81 \Rightarrow 81 = \frac{6}{2}(2a + 5d)$ ***M1A1***

$\Rightarrow 27 = 2a + 5d$

$S_{11} = 231 \Rightarrow 231 = \frac{11}{2}(2a + 10d)$ ***M1A1***

$\Rightarrow 21 = a + 5d$

solving simultaneously, $a = 6$, $d = 3$ ***A1A1***

[6 marks]

(b) $a + ar = 1$ ***A1***

$a + ar + ar^2 + ar^3 = 5$ ***A1***

$\Rightarrow (a + ar) + ar^2(1 + r) = 5$

$\Rightarrow 1 + ar^2 \times \frac{1}{a} = 5$

obtaining $r^2 - 4 = 0$ ***M1***

$\Rightarrow r = \pm 2$

$r = 2$ (since all terms are positive) ***A1***

$a = \frac{1}{3}$ ***A1***

[5 marks]

(c) AP r^{th} term is $3r + 3$ ***A1***

GP r^{th} term is $\frac{1}{3}2^{r-1}$ ***A1***

$3(r + 1) \times \frac{1}{3}2^{r-1} = (r + 1)2^{r-1}$ ***M1AG***

[3 marks]

Total [14 marks]

d. prove: $P_n : \sum\limits_{r=1}^{n} (r + 1)2^{r-1} = n2^n,\ n \in \mathbb{Z}^+$.

show true for $n = 1$, *i.e.*

$\text{LHS} = 2 \times 2^0 = 2 = \text{RHS}$ ***A1***

assume true for $n = k$, *i.e.* ***M1***

$\sum\limits_{r=1}^{k} (r + 1)2^{r-1} = k2^k,\ k \in \mathbb{Z}^+$

consider $n = k + 1$

$\sum\limits_{r=1}^{k+1} (r + 1)2^{r-1} = k2^k + (k + 2)2^k$ ***M1A1***

$= 2^k(k + k + 2)$

$= 2(k + 1)2^k$ ***A1***

$= (k + 1)2^{k+1}$ ***A1***

hence true for $n = k + 1$

P_{k+1} is true whenever P_k is true, and P_1 is true, therefore P_n is true ***R1*** for $n \in \mathbb{Z}^+$

[7 marks]

Examiners report

. Parts (a), (b) and (c) were answered successfully by a large number of candidates. Some, however, had difficulty with the arithmetic.

d. In part (d) many candidates showed little understanding of sigma notation and proof by induction. There were cases of circular reasoning and using *n*, *k* and *r* randomly. A concluding sentence almost always appeared, even if the proof was done incorrectly, or not done at all.

The 1st, 4th and 8th terms of an arithmetic sequence, with common difference d, $d \neq 0$, are the first three terms of a geometric sequence, with common ratio r. Given that the 1st term of both sequences is 9 find

a. the value of d; [4]

b. the value of r; [1]

Markscheme

a. **EITHER**

the first three terms of the geometric sequence are 9, $9r$ and $9r^2$ ***(M1)***

$9 + 3d = 9r (\Rightarrow 3 + d = 3r)$ and $9 + 7d = 9r^2$ ***(A1)***

attempt to solve simultaneously ***(M1)***

$9 + 7d = 9\left(\frac{3+d}{3}\right)^2$

OR

the 1^{st}, 4^{th} and 8^{th} terms of the arithmetic sequence are

$9,\ 9 + 3d,\ 9 + 7d$ ***(M1)***

$\frac{9+7d}{9+3d} = \frac{9+3d}{9}$ ***(A1)***

attempt to solve ***(M1)***

THEN

$d = 1$ ***A1***

[4 marks]

b. $r = \frac{4}{3}$ ***A1***

Note: Accept answers where a candidate obtains d by finding r first. The first two marks in either method for part (a) are awarded for the same ideas and the third mark is awarded for attempting to solve an equation in r.

[1 mark]

Examiners report

a. [N/A]
b. [N/A]

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a. Show that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. [1]

b. Consider $f(x) = \sin(ax)$ where a is a constant. Prove by mathematical induction that $f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$ where $n \in \mathbb{Z}^+$ and $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. [7]

Markscheme

a. $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

b. consider $n = 1,\ f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{\mathrm{d}\left(f^{(k)}(x)\right)}{\mathrm{d}x} \quad \left(= a\left(a^k\cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior ***M*** and ***R*** marks have been awarded.

[7 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

Solve the equation $2 - \log_3(x + 7) = \log_{\frac{1}{3}} 2x$.

Markscheme

$\log_3\left(\frac{9}{x+7}\right) = \log_3 \frac{1}{2x}$ ***M1M1A1***

Note: Award ***M1*** for changing to single base, ***M1*** for incorporating the 2 into a log and ***A1*** for a correct equation with maximum one log expression each side.

$x + 7 = 18x$ ***M1***

$x = \frac{7}{17}$ ***A1***

[5 marks]

Examiners report

Some good solutions to this question and few candidates failed to earn marks on the question. Many were able to change the base of the logs, and many were able to deal with the 2, but of those who managed both, poor algebraic skills were often evident. Many students attempted to change the base into base 10, resulting in some complicated algebra, few of which managed to complete successfully.

Solve the equation $4^x + 2^{x+2} = 3$.

Markscheme

attempt to form a quadratic in 2^x ***M1***

$(2^x)^2 + 4 \bullet 2^x - 3 = 0$ ***A1***

$2^x = \frac{-4 \pm \sqrt{16+12}}{2} \left(= -2 \pm \sqrt{7}\right)$ ***M1***

$2^x = -2 + \sqrt{7} \left(\text{as } -2 - \sqrt{7} < 0\right)$ ***R1***

$x = \log_2\left(-2 + \sqrt{7}\right) \left(x = \frac{\ln\left(-2+\sqrt{7}\right)}{\ln 2}\right)$ ***A1***

Note: Award ***R0 A1*** if final answer is $x = \log_2\left(-2 + \sqrt{7}\right)$.

[5 marks]

Examiners report

[N/A]

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a. Write down and simplify the expansion of $(2 + x)^4$ in ascending powers of x. [3]

b. Hence find the exact value of $(2.1)^4$. [3]

Markscheme

a. $(2+x)^4 = 2^4 + 4 \bullet 2^3 x + 6 \bullet 2^2 x^2 + 4 \bullet 2x^3 + x^4$ ***M1(A1)***

Note: Award ***M1*** for an expansion, by whatever method, giving five terms in any order.

$= 16 + 32x + 24x^2 + 8x^3 + x^4$ ***A1***

Note: Award ***M1A1A0*** for correct expansion not given in ascending powers of x.

[3 marks]

b. let $x = 0.1$ (in the binomial expansion) ***(M1)***

$2.1^4 = 16 + 3.2 + 0.24 + 0.008 + 0.0001$ ***(A1)***

$= 19.4481$ ***A1***

Note: At most one of the marks can be implied.

[3 marks]

Total [6 marks]

Examiners report

a. [N/A]

b. [N/A]

A geometric sequence has first term a, common ratio r and sum to infinity 76. A second geometric sequence has first term a, common ratio r^3 and sum to infinity 36.

Find r.

Markscheme

for the first series $\frac{a}{1-r} = 76$ ***A1***

for the second series $\frac{a}{1-r^3} = 36$ ***A1***

attempt to eliminate a *e.g.* $\frac{76(1-r)}{1-r^3} = 36$ ***M1***

simplify and obtain $9r^2 + 9r - 10 = 0$ ***(M1)A1***

Note: Only award the ***M1*** if a quadratic is seen.

obtain $r = \frac{12}{18}$ and $-\frac{30}{18}$ ***(A1)***

$r = \frac{12}{18}\left(= \frac{2}{3} = 0.666\ldots\right)$ ***A1***

Note: Award ***A0*** if the extra value of *r* is given in the final answer.

Total [7 marks]

Examiners report

Almost all candidates obtained the cubic equation satisfied by the common ratio of the first sequence, but few were able to find its roots. One of the roots was $r = 1$.

The sum of the first two terms of a geometric series is 10 and the sum of the first four terms is 30.

(a) Show that the common ratio r satisfies $r^2 = 2$.

(b) Given $r = \sqrt{2}$

(i) find the first term;

(ii) find the sum of the first ten terms.

Markscheme

(a) **METHOD 1**

$a + ar = 10$ ***A1***

$a + ar + ar^2 + ar^3 = 30$ ***A1***

$a + ar = 10 \Rightarrow ar^2 + ar^3 = 10r^2$ **or** $ar^2 + ar^3 = 20$ ***M1***

$10 + 10r^2 = 30$ **or** $r^2(a + ar) = 20$ ***A1***

$\Rightarrow r^2 = 2$ ***AG***

METHOD 2

$\frac{a(1-r^2)}{1-r} = 10$ and $\frac{a(1-r^4)}{1-r} = 30$ ***M1A1***

$\Rightarrow \frac{1-r^4}{1-r^2} = 3$ ***M1***

leading to either $1 + r^2 = 3$ (or $r^4 - 3r^2 + 2 = 0$) ***A1***

$\Rightarrow r^2 = 2$ ***AG***

[4 marks]

(b) (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \frac{10}{1+\sqrt{2}}$ **or** $a = 10\left(\sqrt{2} - 1\right)$ ***A1***

(ii) $S_{10} = \frac{10}{1+\sqrt{2}}\left(\frac{\sqrt{2}^{10}-1}{\sqrt{2}-1}\right)$ $(= 10 \times 31)$ ***M1***

$= 310$ ***A1***

[3 marks]

Total [7 marks]

Examiners report

This question was invariably answered very well. Candidates showed some skill in algebraic manipulation to derive the given answer in part a). Poor attempts at part b) were a rarity, though the final mark was sometimes lost after a correctly substituted equation was seen but with little follow-up work.

Let $z = \cos\theta + i\sin\theta$.

a. Use de Moivre's theorem to find the value of $\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3$. [2]

b. Use mathematical induction to prove that [6]

$$(\cos\theta - \mathrm{i}\sin\theta)^n = \cos n\theta - \mathrm{i}\sin n\theta \text{ for } n \in \mathbb{Z}^+.$$

c. Find an expression in terms of θ for $(z)^n + (z^*)^n,\ n \in \mathbb{Z}^+$ where z^* is the complex conjugate of z. [2]

d. (i) Show that $zz^* = 1$. [5]

(ii) Write down the binomial expansion of $(z + z^*)^3$ in terms of z and z^*.

(iii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

e. Hence solve $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \leqslant \theta < \pi$. [6]

Markscheme

a. $\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)^3 = \cos\pi + \mathrm{i}\sin\pi$ ***M1***

$= -1$ ***A1***

[2 marks]

b. show the expression is true for $n = 1$ ***R1***

assume true for $n = k,\ (\cos\theta - \mathrm{i}\sin\theta)^k = \cos k\theta - \mathrm{i}\sin k\theta$ ***M1***

Note: Do not accept "let $n = k$" or "assume $n = k$", assumption of truth must be present.

$(\cos\theta - \mathrm{i}\sin\theta)^{k+1} = (\cos\theta - \mathrm{i}\sin\theta)^k(\cos\theta - \mathrm{i}\sin\theta)$

$= (\cos k\theta - \mathrm{i}\sin k\theta)(\cos\theta - \mathrm{i}\sin\theta)$ ***M1***

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta - \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$ ***A1***

Note: Award ***A1*** for any correct expansion.

$= \cos((k+1)\theta) - \mathrm{i}\sin((k+1)\theta)$ ***A1***

therefore if true for $n = k$ true for $n = k + 1$, true for $n = 1$, so true for all $n(\in \mathbb{Z}^+)$ ***R1***

Note: To award the final ***R*** mark the first 4 marks must be awarded.

[6 marks]

c. $(z)^n + (z^*)^n = (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta - \mathrm{i}\sin\theta)^n$

$= \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta = 2\cos(n\theta)$ ***(M1)A1***

[2 marks]

d. (i) $zz* = (\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)$

$= \cos^2\theta + \sin^2\theta$ ***A1***

$= 1$ ***AG***

Note: Allow justification starting with $|z| = 1$.

(ii) $(z + z^*)^3 = z^3 + 3z^2z^* + 3z(z^*)^2 + (z^*)3 \left(= z^3 + 3z + 3z^* + (z^*)^3\right)$ ***A1***

(iii) $(z + z^*)^3 = (2\cos\theta)^3$ ***A1***

$z^3 + 3z + 3z^* + (z^*)^3 = 2\cos 3\theta + 6\cos\theta$ ***M1A1***

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ***AG***

Note: ***M1*** is for using $zz^* = 1$, this might be seen in d(ii).

[5 marks]

e. $4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$\cos(3\theta) = \cos(2\theta)$ ***A1A1***

Note: ***A1*** for $\cos(3\theta)$ and ***A1*** for $\cos(2\theta)$.

$\theta = 0$ ***A1***

or $3\theta = 2\pi - 2\theta$ $(\text{or } 3\theta = 4\pi - 2\theta)$ ***M1***

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}$ ***A1A1***

Note: Do not accept solutions via factor theorem or other methods that do not follow “hence”.

[6 marks]

Examiners report

a. This was well done by most candidates who correctly applied de Moivre’s theorem.

b. This question was poorly done, which was surprising as it is very similar to the proof of de Moivre’s theorem which is stated as being required in the course guide. Many candidates spotted that they needed to use trigonometric identities but fell down through not being able to set out the proof in a logical form.

c. This was well done by the majority of candidates.

d. (d) parts (i) and (ii) were well done by the candidates, who were able to successfully use trigonometrical identities and the binomial theorem.

(d)(iii) This is a familiar technique that has appeared in several recent past papers and was successfully completed by many of the better candidates. Some candidates though neglected the instruction ‘hence’ and tried to derive the expression using trigonometric identities.

e. Again some candidates ignored ‘hence’ and tried to form a polynomial equation. Many candidates obtained the solution $\cos(2\theta) = \cos(3\theta)$ and hence the solution $\theta = 0$. Few were able to find the other solutions which can be obtained from consideration of the unit circle or similar methods.

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a. (i) Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus-argument form. [9]

(ii) Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

(iii) Show that $z_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$.

b. (i) State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus-argument form. [9]

(ii) If *w* is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

(iii) Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients.

Markscheme

a. (i) $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right),\ z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right),\ z_3 = 2\text{cis}\left(-\frac{\pi}{2}\right)$ or $2\text{cis}\left(\frac{3\pi}{2}\right)$ ***A1A1A1***

Note: Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note: Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin ***A1***

differences are all $\frac{2\pi}{3}(\bmod 2\pi)$ ***A1***

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral ***R1AG***

Note: Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\text{cis}\left(\frac{5n\pi}{2}\right)$ ***M1***

$= 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1***

$2z_3^{3n} = 2 \times 2^{3n}\text{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1AG***

[9 marks]

b. (i) attempt to obtain **seven** solutions in modulus argument form ***M1***

$z = \text{cis}\left(\frac{2k\pi}{7}\right),\ k = 0,\ 1\ldots 6$ ***A1***

(ii) *w* has argument $\frac{2\pi}{7}$ and $1 + w$ has argument ϕ,

then $\tan(\phi) = \frac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}$ ***M1***

$= \frac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$ ***A1***

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ ***A1***

Note: Accept alternative approaches.

(iii) since roots occur in conjugate pairs, ***(R1)***

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ ***A1***

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ ***AG***

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ ***A1*** and $z^2-2z\cos\left(\frac{6\pi}{7}\right)+1$ ***A1***

[9 marks]

Examiners report

a. (i) A disappointingly large number of candidates were unable to give the correct arguments for the three complex numbers. Such errors undermined their efforts to tackle parts (ii) and (iii).

b. Many candidates were successful in part (i), but failed to capitalise on that – in particular, few used the fact that roots of $z^7-1=0$ come in complex conjugate pairs.

Consider the complex numbers $z_1=2\text{cis}150^\circ$ and $z_2=-1+\text{i}$.

a. Calculate $\frac{z_1}{z_2}$ giving your answer both in modulus-argument form and Cartesian form. [7]

b. Using your results, find the exact value of tan 75° , giving your answer in the form $a+\sqrt{b}$, a , $b\in\mathbb{Z}^+$. [5]

Markscheme

a. in Cartesian form

$z_1=2\times-\frac{\sqrt{3}}{2}+2\times\frac{1}{2}\text{i}$ ***M1***

$=-\sqrt{3}+\text{i}$ ***A1***

$\frac{z_1}{z_2}=\frac{-\sqrt{3}+\text{i}}{-1+\text{i}}$

$=\frac{\left(-\sqrt{3}+\text{i}\right)}{(-1+\text{i})}\times\frac{(-1-\text{i})}{(-1-\text{i})}$ ***M1***

$=\frac{1+\sqrt{3}}{2}+\frac{\left(\sqrt{3}-1\right)}{2}\text{i}$ ***A1***

in modulus-argument form

$\text{z}_2=\sqrt{2}\text{cis}135^\circ$ ***A1***

$\frac{z_1}{z_2}=\frac{2\text{cis}150^\circ}{\sqrt{2}\text{cis}135^\circ}$

$=\sqrt{2}\text{cis}15^\circ$ ***A1A1***

[7 marks]

b. equating the two expressions for $\frac{z_1}{z_2}$

$\cos 15^\circ=\frac{1+\sqrt{3}}{2\sqrt{2}}$ ***A1***

$\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$ ***A1***

$\tan 75^\circ=\frac{\cos 15^\circ}{\sin 15^\circ}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ ***M1***

$=\frac{\left(\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$ ***A1***

$=2+\sqrt{3}$ ***A1***

[5 marks]

Examiners report

a. [N/A]
b. [N/A]

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a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. [2]

b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. [2]

c. Prove, by mathematical induction, that $\sum\limits_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$. [9]

Markscheme

a. $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1} - \sqrt{n}$ ***AG***

[2 marks]

b. $\sqrt{2} - 1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

c. consider the case $n = 2$: required to prove that $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2} - 1$

hence $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k : \sum\limits_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k + 1 : \frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

from assumption, we have that $\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k}+\frac{1}{\sqrt{k+1}}=\frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$ ***A1***

$>\frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}}=\sqrt{k+1}$ ***A1***

THEN

so true for $n=2$ and $n=k$ true $\Rightarrow n=k+1$ true. Hence true for all $n\geq 2$ ***R1***

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Prove by mathematical induction $\sum\limits_{r=1}^{n} r(r!)=(n+1)!-1, n\in\mathbb{Z}^{+}$.

Markscheme

let $n=1$

$\text{LHS}=1\times 1!=1$

$\text{RHS}=(1+1)!-1=2-1=1$

hence true for $n=1$ ***R1***

assume true for $n=k$

$\sum\limits_{r=1}^{k} r(r!)=(k+1)!-1$ ***M1***

$\sum\limits_{r=1}^{k+1} r(r!)=(k+1)!-1+(k+1)\times(k+1)!$ ***M1A1***

$=(k+1)!(1+k+1)-1$

$=(k+1)!(k+2)-1$ ***A1***

$=(k+2)!-1$ ***A1***

hence if true for $n=k$, true for $n=k+1$ ***R1***

since the result is true for $n=1$ and $\mathrm{P}(k)\Rightarrow\mathrm{P}(k+1)$ the result is proved by mathematical induction $\forall n\in\mathbb{Z}^{+}$ ***R1***

[8 marks]

Examiners report

This question was done poorly on a number of levels. Many students knew the structure of induction but did not show that they understood what they were doing. The general notation was poor for both the induction itself and the sigma notation.

In noting the case for $n = 1$ too many stated the equation rather than using the LHS and RHS separately and concluding with a statement. There were also too many who did not state the conclusion for this case.

Many did not state the assumption for $n = k$ as an assumption.

Most stated the equation for $n = k + 1$ and worked with the equation. Also common was the lack of sigma and inappropriate use of n and k in the statement. There were some very nice solutions however.

The final conclusion was often not complete or not considered which would lead to the conclusion that the student did not really understand what induction is about.

The cubic equation $x^3 + px^2 + qx + c = 0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that

a. (i) $p = -(\alpha + \beta + \gamma)$; [3]

(ii) $q = \alpha\beta + \beta\gamma + \gamma\alpha$;

(iii) $c = -\alpha\beta\gamma$.

b. It is now given that $p = -6$ and $q = 18$ for parts (b) and (c) below. [5]

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. Determine the value of c. [6]

Markscheme

a. (i)-(iii) given the three roots $\alpha,\ \beta,\ \gamma$, we have

$x^3 + px^2 + qx + c = (x - \alpha)(x - \beta)(x - \gamma)$ ***M1***

$= \left(x^2 - (\alpha + \beta)x + \alpha\beta\right)(x - \gamma)$ ***A1***

$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha + \beta + \gamma)$ ***AG***

$q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

b. **METHOD 1**

(i) Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$

So $\beta - \alpha = \gamma - \beta$ ***M1***

or $2\beta = \alpha + \gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta + 2\beta = 6$ ***A1***

$\beta = 2$ ***AG***

(ii) $\alpha + \gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \frac{4 \pm i\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{4+i\sqrt{24}}{2}\right)\left(\frac{4-i\sqrt{24}}{2}\right)2 = -20$ ***A1***

METHOD 2

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$

$c = -20$ ***A1***

METHOD 3

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}i$

$\Rightarrow c = -20$ ***A1***

[5 marks]

c. **METHOD 1**

Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$.

So $\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$ ***M1***

or $\beta^2 = \alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta + \gamma\beta + \beta^2 = 18$

$\beta(\alpha + \beta + \gamma) = 18$

$6\beta = 18$

$\beta = 3$ ***A1***

$\alpha + \gamma = 3,\ \alpha = \frac{9}{\gamma}$

$\Rightarrow \gamma^2 - 3\gamma + 9 = 0$

$\Rightarrow \gamma = \frac{3 \pm i\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{3+i\sqrt{27}}{2}\right)\left(\frac{3-i\sqrt{27}}{2}\right)3 = -27$ ***A1***

METHOD 2

let the three roots be $a,\ ar,\ ar^2$ ***M1***

attempt at substitution of $a,\ ar,\ ar^2$ and p and q into equations from (a) ***M1***

$6 = a + ar + ar^2\ \left(= a(1 + r + r^2)\right)$ ***A1***

$18 = a^2r + a^2r^3 + a^2r^2\ \left(= a^2r(1 + r + r^2)\right)$ ***A1***

therefore $3 = ar$ ***A1***

therefore $c = -a^3r^3 = -3^3 = -27$ ***A1***

[6 marks]

Total [14 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Let $\{u_n\},\ n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\},\ n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

a. (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant. [4]

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n.

b. Let S_n be the sum of the first n terms of the sequence $\{v_n\}$. [8]

(i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum\limits_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum\limits_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d .

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d .

c. Let $\{w_n\},\ n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$. [6]

Find $\sum\limits_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q.

Markscheme

a. (i) **METHOD 1**

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ ***M1***

$= 2^{u_{n+1}-u_n} = 2^d$ ***A1***

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ ***M1***

$= 2^d$ ***A1***

(ii) $= 2^a$ ***A1***

Note: Accept $= 2^{u_1}$.

(iii) **EITHER**

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a(2^d)^{(n-1)}$

OR

$u_n = a + (n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ ***A1***

[4 marks]

b. (i) $S_n = \frac{2^a\left(\left(2^d\right)^n-1\right)}{2^d-1} = \frac{2^a(2^{dn}-1)}{2^d-1}$ ***M1A1***

Note: Accept either expression.

(ii) for sum to infinity to exist need $-1 < 2^d < 1$ ***R1***

$\Rightarrow \log 2^d < 0 \Rightarrow d\log 2 < 0 \Rightarrow d < 0$ ***(M1)A1***

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1-2^d}$ ***A1***

(iv) $\frac{2^a}{1-2^d} = 2^{a+1} \Rightarrow \frac{1}{1-2^d} = 2$ ***M1***

$\Rightarrow 1 = 2 - 2^{d+1} \Rightarrow 2^{d+1} = 1$

$\Rightarrow d = -1$ ***A1***

[8 marks]

c. **METHOD 1**

$w_n = pq^{n-1},\ z_n = \ln pq^{n-1}$ ***(A1)***

$z_n = \ln p + (n-1)\ln q$ ***M1A1***

$z_{n+1} - z_n = (\ln p + n\ln q) - (\ln p + (n-1)\ln q) = \ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum\limits_{i=1}^{n} z_i = \frac{n}{2}(2\ln p + (n-1)\ln q)$ ***M1***

$= n\left(\ln p + \ln q^{\left(\frac{n-1}{2}\right)}\right) = n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ ***(M1)***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***A1***

METHOD 2

$\sum\limits_{i=1}^{n} z_i = \ln p + \ln pq + \ln pq^2 + \ldots + \ln pq^{n-1}$ ***(M1)A1***

$= \ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ ***(M1)A1***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***(M1)A1***

[6 marks]

Total [18 marks]

Examiners report

a. Method of first part was fine but then some algebra mistakes often happened. The next two parts were generally good.

b. Given that (a) indicated that there was a common ratio a disappointing number thought it was an AP. Although some good answers in the next parts, there was also some poor notational misunderstanding with the sum to infinity still involving n.

c. Not enough candidates realised that this was an AP.

Express $\frac{1}{(1-\mathrm{i}\sqrt{3})^3}$ in the form $\frac{a}{b}$ where $a,\ b \in \mathbb{Z}$.

Markscheme

METHOD 1

$r = 2,\ \theta = -\frac{\pi}{3}$ ***(A1)(A1)***

$\therefore (1-\mathrm{i}\sqrt{3})^{-3} = 2^{-3}\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)^{-3}$ ***M1***

$= \frac{1}{8}(\cos\pi + \mathrm{i}\sin\pi)$ ***(M1)***

$= -\frac{1}{8}$ ***A1***

[5 marks]

METHOD 2

$(1-\sqrt{3}\mathrm{i})(1-\sqrt{3}\mathrm{i}) = 1 - 2\sqrt{3}\mathrm{i} - 3 \quad (= -2 - 2\sqrt{3}\mathrm{i})$ ***(M1)A1***

$(-2-2\sqrt{3}\mathrm{i})(1-\sqrt{3}\mathrm{i}) = -8$ ***(M1)(A1)***

$\therefore \frac{1}{(1-\sqrt{3}\mathrm{i})^3} = -\frac{1}{8}$ ***A1***

[5 marks]

METHOD 3

Attempt at Binomial expansion ***M1***

$(1-\sqrt{3}\mathrm{i})^3 = 1 + 3(-\sqrt{3}\mathrm{i}) + 3(-\sqrt{3}\mathrm{i})^2 + (-\sqrt{3}\mathrm{i})^3$ ***(A1)***

$= 1 - 3\sqrt{3}\mathrm{i} - 9 + 3\sqrt{3}\mathrm{i}$ ***(A1)***

$= -8$ ***A1***

$\therefore \frac{1}{(1-\sqrt{3}\mathrm{i})^3} = -\frac{1}{8}$ ***M1***

[5 marks]

Examiners report

Most candidates made a meaningful attempt at this question using a variety of different, but correct methods. Weaker candidates sometimes made errors with the manipulation of the square roots, but there were many fully correct solutions.

Find the value of k if $\sum\limits_{r=1}^{\infty} k\left(\frac{1}{3}\right)^r = 7$.

Markscheme

$u_1 = \frac{1}{3}k$, $r = \frac{1}{3}$ ***(A1) (A1)***

$7 = \frac{\frac{1}{3}k}{1-\frac{1}{3}}$ ***M1***

$k = 14$ ***A1***

[4 marks]

Examiners report

The question was well done generally. Those that did make mistakes on the question usually had the first term wrong, but did understand to use the formula for an infinite geometric series.

Given that $z_1 = 2$ and $z_2 = 1 + \sqrt{3}\mathrm{i}$ are roots of the cubic equation $z^3 + bz^2 + cz + d = 0$

where b, c, $d \in \mathbb{R}$,

(a) write down the third root, z_3, of the equation;

(b) find the values of b, c and d ;

(c) write z_2 and z_3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

Markscheme

(a) $1 - \sqrt{3}\mathrm{i}$ ***A1***

(b) **EITHER**

$\left(z - (1+\sqrt{3}\mathrm{i})\right)\left(z - (1-\sqrt{3}\mathrm{i})\right) = z^2 - 2z + 4$ ***(M1)A1***

$p(z) = (z-2)(z^2 - 2z + 4)$ ***(M1)***

$= z^3 - 4z^2 + 8z - 8$ ***A1***

therefore $b = -4,\ c = 8,\ d = -8$

OR

relating coefficients of cubic equations to roots

$-b = 2 + 1 + \sqrt{3}\mathrm{i} + 1 - \sqrt{3}\mathrm{i} = 4$ ***M1***

$c = 2(1 + \sqrt{3}\mathrm{i}) + 2(1 - \sqrt{3}\mathrm{i}) + (1 + \sqrt{3}\mathrm{i})(1 - \sqrt{3}\mathrm{i}) = 8$

$-d = 2(1 + \sqrt{3}\mathrm{i})(1 - \sqrt{3}\mathrm{i}) = 8$

$b = -4,\ c = 8,\ d = -8$ ***A1A1A1***

(c) $z_2 = 2\mathrm{e}^{\frac{\mathrm{i}\pi}{3}},\ z_3 = 2\mathrm{e}^{-\frac{\mathrm{i}\pi}{3}}$ ***A1A1A1***

Note: Award ***A1*** for modulus,

A1 for each argument.

[8 marks]

Examiners report

Parts a) and c) were done quite well by many but the method used in b) often lead to tedious and long algebraic manipulations in which students got lost and so did not get to the correct solution. Many did not give the principal argument in c).

Consider the function $f_n(x) = (\cos 2x)(\cos 4x)\ldots(\cos 2^n x),\ n \in \mathbb{Z}^+$.

a. Determine whether f_n is an odd or even function, justifying your answer. [2]

b. By using mathematical induction, prove that [8]

$$f_n(x) = \frac{\sin 2^{n+1}x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2} \text{ where } m \in \mathbb{Z}.$$

c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. [3]

d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. [8]

Markscheme

a. even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x)\ldots = (\cos(-2x))\,(\cos(-4x))\ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

b. consider the case $n = 1$

$\frac{\sin 4x}{2 \sin 2x} = \frac{2 \sin 2x \cos 2x}{2 \sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x) \ldots (\cos 2^k x) = \frac{\sin 2^{k+1} x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1} x)$ ***(M1)***

$= \frac{\sin 2^{k+1} x}{2^k \sin 2x} \cos 2^{k+1} x$ ***A1***

$= \frac{2 \sin 2^{k+1} x \cos 2^{k+1} x}{2^{k+1} \sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2} x}{2^{k+1} \sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

c. attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$f_n'(x) = \frac{(2^n \sin 2x)(2^{n+1} \cos 2^{n+1} x) - (\sin 2^{n+1} x)(2^{n+1} \cos 2x)}{(2^n \sin 2x)^2}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

d. $f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin \frac{\pi}{2}\right)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right) - \left(\sin 2^{n+1} \frac{\pi}{4}\right)\left(2^{n+1} \cos \frac{\pi}{2}\right)}{\left(2^n \sin \frac{\pi}{2}\right)^2}$ ***(M1)(A1)***

$f_n'\left(\frac{\pi}{4}\right) = \frac{(2^n)\left(2^{n+1} \cos 2^{n+1} \frac{\pi}{4}\right)}{(2^n)^2}$ ***(A1)***

$= 2 \cos 2^{n+1} \frac{\pi}{4}$ $(= 2 \cos 2^{n-1} \pi)$ ***A1***

$f_n'\left(\frac{\pi}{4}\right) = 2$ ***A1***

$f_n\left(\frac{\pi}{4}\right) = 0$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$y = 2\left(x - \frac{\pi}{4}\right)$ ***M1A1***

$4x - 2y - \pi = 0$ ***AG***

[8 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in \mathbb{Z}^+$.

Markscheme

let $P(n)$ be the proposition that $4^n + 15n - 1$ is divisible by 9

showing true for $n = 1$ ***A1***

ie for $n = 1,\ 4^1 + 15 \times 1 - 1 = 18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k + 15k - 1 = 9A,\ (A \in \mathbb{Z}^+)$ ***M1***

Note: Only award ***M1*** if “truth assumed” or equivalent.

consider $4^{k+1} + 15(k+1) - 1$

$= 4 \times 4^k + 15k + 14$

$= 4(9A - 15k + 1) + 15k + 14$ ***M1***

$= 4 \times 9A - 45k + 18$ ***A1***

$= 9(4A - 5k + 2)$ which is divisible by 9 ***R1***

Note: Award ***R1*** for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if the 2 ***M1***s have been awarded.

[6 marks]

Examiners report

[N/A]

Expand and simplify $\left(x^2 - \frac{2}{x}\right)^4$.

Markscheme

$\left(x^2 - \frac{2}{x}\right)^4 = (x^2)^4 + 4(x^2)^3\left(-\frac{2}{x}\right) + 6(x^2)^2\left(-\frac{2}{x}\right)^2 + 4(x^2)\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$ ***(M1)***

$= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$ ***A3***

Note: Deduct one ***A*** mark for each incorrect or omitted term.

[4 marks]

Examiners report

Most candidates solved this question correctly with most candidates who explained how they obtained their coefficients using Pascal's triangle rather than the combination formula.

Expand $(3 - x)^4$ in ascending powers of x and simplify your answer.

Markscheme

$(3 - x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent ***(M1)(A1)***

$= 81 - 108x + 54x^2 - 12x^3 + x^4$ ***A1A1***

Note: ***A1*** for ascending powers, ***A1*** for correct coefficients including signs.

[4 marks]

Examiners report

[N/A]

(a) Consider the following sequence of equations.

$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3)$,

$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4)$,

$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5)$,

$\dots$.

(i) Formulate a conjecture for the n^{th} equation in the sequence.

(ii) Verify your conjecture for $n = 4$.

(b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3,\ n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(c) Use mathematical induction to prove that $5 \times 7^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

Markscheme

(a) (i) $1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ ***R1***

(ii) LHS = 40; RHS = 40 ***A1***

[2 marks]

(b) the sequence of values are:

5, 7, 11, 19, 35 … or an example ***A1***

35 is not prime, so Bill's conjecture is false ***R1AG***

[2 marks]

(c) $\mathrm{P}(n) : 5 \times 7^n + 1$ is divisible by 6

$\mathrm{P}(1) : 36$ is divisible by $6 \Rightarrow \mathrm{P}(1)$ true ***A1***

assume $\mathrm{P}(k)$ is true $(5 \times 7^k + 1 = 6r)$ ***M1***

Note: Do **not** award ***M1*** for statement starting 'let $n = k$'.

Subsequent marks are independent of this ***M1***.

consider $5 \times 7^{k+1} + 1$ ***M1***

$= 7(6r - 1) + 1$ ***(A1)***

$= 6(7r - 1) \Rightarrow \mathrm{P}(k+1)$ is true ***A1***

P(1) true and $\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, so by MI $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if there is consideration of P(1), $\mathrm{P}(k)$ and $\mathrm{P}(k+1)$ in the final statement.

Only award ***R1*** if at least one of the two preceding ***A*** marks has been awarded.

[6 marks]

Total [10 marks]

Examiners report

Although there were a good number of wholly correct solutions to this question, it was clear that a number of students had not been prepared for questions on conjectures. The proof by induction was relatively well done, but candidates often showed a lack of rigour in the proof. It was fairly common to see students who did not appreciate the idea that $\mathrm{P}(k)$ is assumed not given and this was penalised. Also it appeared that a number of students had been taught to write down the final reasoning for a proof by induction, even if no attempt of a proof had taken place. In these cases, the final reasoning mark was not awarded.

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a. (i) Show that $(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}, \quad \cos\theta \neq 0.$ [10]

(ii) Hence verify that $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation $(1+z)^4 + (1-z)^4 = 0, \quad z \in \mathbb{C}.$

(iii) State another root of the equation $(1+z)^4 + (1-z)^4 = 0, \quad z \in \mathbb{C}.$

b. (i) Use the double angle identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ to show that $\tan\frac{\pi}{8} = \sqrt{2} - 1.$ [13]

(ii) Show that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1.$

(iii) Hence find the value of $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x.$

Markscheme

a. (i) **METHOD 1**

$(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = \left(1+\mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n + \left(1-\mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^n$ ***M1***

$= \left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem ***(M1)***

$\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$ ***A1***

recognition that $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$ ***(R1)***

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta-i\sin n\theta}{\cos^n\theta}$ ***A1***

$(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

METHOD 2

$(1+\mathrm{i}\tan\theta)^n + (1-\mathrm{i}\tan\theta)^n = (1+\mathrm{i}\tan\theta)^n + (1+\mathrm{i}\tan(-\theta))^n$ ***(M1)***

$= \frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta} + \frac{(\cos(-\theta)+i\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$= \frac{1}{\cos^n\theta}(\cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta))$ ***A1***

$= \frac{2\cos n\theta}{\cos^2\theta} \quad \text{as} \quad \cos(-n\theta) = \cos n\theta \quad \text{and} \quad \sin(-n\theta) = -\sin n\theta$ ***R1AG***

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4 + \left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4 = \frac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$ ***(A1)***

$= \frac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$ ***A1***

$= 0 \quad \text{as} \quad \cos\frac{3\pi}{2} = 0$ ***R1***

Note: The above working could involve theta and the solution of $\cos(4\theta) = 0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation ***AG***

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$ ***A1***

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.

Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

[10 marks]

b. (i) $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$ ***A1***

let $t = \tan\frac{\pi}{8}$

attempting to solve $t^2 + 2t - 1 = 0$ for t ***M1***

$t = -1 \pm \sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8} > 0$ ***R1***

$\tan\frac{\pi}{8} = \sqrt{2} - 1$ ***AG***

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\mathrm{d}x$ ***A1***

$= [4\sin 2x - 8x + 2\tan x]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]

Examiners report

a. Fairly successful.

b. (i) Most candidates attempted to use the hint. Those who doubled the angle were usually successful – but many lost the final mark by not giving a convincing reason to reject the negative solution to the intermediate quadratic equation. Those who halved the angle got nowhere.

(ii) The majority of candidates obtained full marks.

(iii) This was poorly answered, few candidates realising that part of the integrand could be re-expressed using $\frac{1}{\cos^2 x} = \sec^2 x$, which can be immediately integrated.

Consider the equation $9x^3 - 45x^2 + 74x - 40 = 0$.

a. Write down the numerical value of the sum and of the product of the roots of this equation. [1]

b. The roots of this equation are three consecutive terms of an arithmetic sequence. [6]

Taking the roots to be α , $\alpha \pm \beta$, solve the equation.

Markscheme

a. sum $= \frac{45}{9}$, product $= \frac{40}{9}$ ***A1***

[1 mark]

b. it follows that $3\alpha = \frac{45}{9}$ and $\alpha(\alpha^2 - \beta^2) = \frac{40}{9}$ ***A1A1***

solving, $\alpha = \frac{5}{3}$ ***A1***

$\frac{5}{3}\left(\frac{25}{9} - \beta^2\right) = \frac{40}{9}$ ***M1***

$\beta = (\pm)\frac{1}{3}$ ***A1***

the other two roots are 2, $\frac{4}{3}$ ***A1***

[6 marks]

Examiners report

a. [N/A]

b. [N/A]

(a) Show that the following system of equations has an infinite number of solutions.

$x + y + 2z = -2$

$3x - y + 14z = 6$

$x + 2y = -5$

The system of equations represents three planes in space.

(b) Find the parametric equations of the line of intersection of the three planes.

Markscheme

(a) **EITHER**

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \textbf{\textit{M1}}$$

row of zeroes implies infinite solutions, (or equivalent). ***R1***

Note: Award ***M1*** for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \quad \textbf{\textit{M1}}$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ with one valid point} \quad \textbf{\textit{R1}}$$

OR

$x + y + 2z = -2$

$3x - y + 14z = 6$

$x + 2y = -5 \ \Rightarrow x = -5 - 2y$

substitute $x = -5 - 2y$ into the first two equations:

$-5 - 2y + y + 2z = -2$

$3(-5 - 2y) - y + 14z = 6$ ***M1***

$-y + 2z = 3$

$-7y + 14z = 21$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions. ***R1***

OR

for example, $7 \times \mathrm{R}_1 - \mathrm{R}_2$ gives $4x + 8y = -20$ ***M1***

this equation is a multiple of the third equation, therefore an infinite number of solutions. ***R1***

[2 marks]

(b) let $y = t$ ***M1***

then $x = -5 - 2t$ ***A1***

$z = \frac{t+3}{2}$ ***A1***

OR

let $x = t$ ***M1***

then $y = \frac{-5-t}{2}$ ***A1***

$z = \frac{1-t}{4}$ ***A1***

OR

let $z = t$ ***M1***

then $x = 1 - 4t$ ***A1***

$y = -3 + 2t$ ***A1***

OR

attempt to find cross product of two normal vectors:

eg: $\begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k$ ***M1A1***

$x = 1 - 4t$
$y = -3 + 2t$
$z = t$ ***A1***
(or equivalent)
[3 marks]

Total [5 marks]

Examiners report

[N/A]

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+,\ \sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2},\ 0 < x < \pi$.

Markscheme

(a) $\sin(2n+1)x\cos x - \cos(2n+1)x\sin x = \sin(2n+1)x - x$ ***M1A1***

$= \sin 2nx$ ***AG***

[2 marks]

(b) if $n = 1$ ***M1***

$\text{LHS} = \cos x$

$\text{RHS} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$ ***M1***

so LHS = RHS and the statement is true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

Note: Only award ***M1*** if the word **true** appears.

Do **not** award ***M1*** for 'let $n = k$' only.

Subsequent marks are independent of this ***M1***.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \frac{\sin 2kx}{2\sin x}$

if $n = k + 1$ then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ ***M1***

$= \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x$ ***A1***

$= \frac{\sin 2kx + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x}$ ***M1***

$= \frac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x}$ ***A1***

$= \frac{\sin(2k+2)x}{2\sin x}$ ***M1***

$= \frac{\sin 2(k+1)x}{2\sin x}$ ***A1***

so if true for $n = k$, then also true for $n = k + 1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Final ***R1*** is independent of previous work.

[12 marks]

(c) $\frac{\sin 4x}{2\sin x} = \frac{1}{2}$ ***M1A1***

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \frac{\pi}{5}$ ***A1***

$4x = 2\pi + x \Rightarrow x = \frac{2\pi}{3}$ ***A1***

$4x = 3\pi - x \Rightarrow x = \frac{3\pi}{5}$ ***A1***

for not including any answers outside the domain ***R1***

Note: Award the first ***M1A1*** for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\left(-\frac{1}{2}, \frac{1 \pm \sqrt{5}}{4}\right)$.

[6 marks]

Total [20 marks]

Examiners report

This question showed the weaknesses of many candidates in dealing with formal proofs and showing their reasoning in a logical manner. In part (a) just a few candidates clearly showed the result and part (b) showed that most candidates struggle with the formality of a proof by induction. The logic of many solutions was poor, though sometimes contained correct trigonometric work. Very few candidates were successful in answering part (c) using the unit circle. Most candidates attempted to manipulate the equation to obtain a cubic equation but made little progress. A few candidates guessed $\frac{2\pi}{3}$ as a solution but were not able to determine the other solutions.

Find the cube roots of i in the form $a + b\mathrm{i}$, where $a,\ b \in \mathbb{R}$.

Markscheme

$\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$ ***(A1)***

$z_1 = \mathrm{i}^{\frac{1}{3}} = \left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)^{\frac{1}{3}} = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} \quad \left(= \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ ***M1A1***

$z_2 = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} \quad \left(= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ ***(M1)A1***

$z_3 = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$ ***A1***

Note: Accept exponential and cis forms for intermediate results, but not the final roots.

Note: Accept the method based on expanding $(\mathrm{a} + \mathrm{b})^3$. ***M1*** for attempt, ***M1*** for equating real and imaginary parts, ***A1*** for finding a = 0 and $\mathrm{b} = \frac{1}{2}$, then ***A1A1A1*** for the roots.

[6 marks]

Examiners report

A varied response. Many knew how to solve this standard question in the most efficient way. A few candidates expanded $(a + ib)^3$ and solved the resulting fairly simple equations. A disappointing minority of candidates did not know how to start.

---

a. Factorize $z^3 + 1$ into a linear and quadratic factor. [2]

b. Let $\gamma = \frac{1+\mathrm{i}\sqrt{3}}{2}$. [9]

(i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

Markscheme

a. using the factor theorem z +1 is a factor ***(M1)***

$z^3 + 1 = (z+1)(z^2 - z + 1)$ ***A1***

[2 marks]

b. (i) **METHOD 1**

$z^3 = -1 \Rightarrow z^3 + 1 = (z+1)(z^2 - z + 1) = 0$ ***(M1)***

solving $z^2 - z + 1 = 0$ ***M1***

$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \mathrm{i}\sqrt{3}}{2}$ ***A1***

therefore one cube root of -1 is γ ***AG***

METHOD 2

$\gamma^2 = \left(\frac{1+i\sqrt{3}}{2}^2\right) = \frac{-1+i\sqrt{3}}{2}$ ***M1A1***

$\gamma^2 = \frac{-1+i\sqrt{3}}{2} \times \frac{1+i\sqrt{3}}{2} = \frac{-1-3}{4}$ ***A1***

$= -1$ AG

METHOD 3

$\gamma = \frac{1+i\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$ ***M1A1***

$\gamma^3 = e^{i\pi} = -1$ ***A1***

(ii) **METHOD 1**

as γ is a root of $z^2 - z + 1 = 0$ then $\gamma^2 - \gamma + 1 = 0$ ***M1R1***

$\therefore \gamma^2 = \gamma - 1$ ***AG***

Note: Award ***M1*** for the use of $z^2 - z + 1 = 0$ in any way.
Award ***R1*** for a correct reasoned approach.
METHOD 2
$\gamma^2 = \frac{-1+i\sqrt{3}}{2}$ ***M1***
$\gamma - 1 = \frac{1+i\sqrt{3}}{2} - 1 = \frac{-1+i\sqrt{3}}{2}$ ***A1***

(iii) **METHOD 1**
$(1-\gamma)^6 = (-\gamma^2)^6$ ***(M1)***
$= (\gamma)^{12}$ ***A1***
$= (\gamma^3)^4$ ***(M1)***
$= (-1)^4$
$= 1$ ***A1***
METHOD 2
$(1-\gamma)^6$
$= 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6$ ***M1A1***
Note: Award ***M1*** for attempt at binomial expansion.

use of any previous result *e.g.* $= 1 - 6\gamma + 15\gamma^2 + 20 - 15\gamma + 6\gamma^2 + 1$ ***M1***
= 1 ***A1***
Note: As the question uses the word 'hence', other methods that do not use previous results are awarded no marks.

[9 marks]

Examiners report

a. In part a) the factorisation was, on the whole, well done.

b. Part (b) was done well by most although using a substitution method rather than the result above. This used much m retime than was necessary but was successful. A number of candidates did not use the previous results in part (iii) and so seemed to not understand the use of the 'hence'.

An arithmetic sequence $u_1, u_2, u_3 \ldots$ has $u_1 = 1$ and common difference $d \neq 0$. Given that u_2, u_3 and u_6 are the first three terms of a geometric sequence

Given that $u_N = -15$

a. find the value of d. [4]

b. determine the value of $\sum_{r=1}^{N} u_r$. [3]

Markscheme

a. use of $u_n = u_1 + (n-1)d$ ***M1***

$(1+2d)^2 = (1+d)(1+5d)$ (or equivalent) ***M1A1***

$d = -2$ ***A1***

[4 marks]

b. $1+(N-1)\times -2=-15$

$N=9$ ***(A1)***

$\sum_{r=1}^{9} u_r = \frac{9}{2}(2+8\times -2)$ ***(M1)***

$=-63$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

If $z_1 = a + a\sqrt{3}i$ and $z_2 = 1 - i$, where a is a real constant, express z_1 and z_2 in the form $r\,\mathrm{cis}\,\theta$, and hence find an expression for $\left(\frac{z_1}{z_2}\right)^6$ in terms of a and i.

Markscheme

$z_1 = 2a\mathrm{cis}\left(\frac{\pi}{3}\right),\ z_2 = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$ ***M1 A1 A1***

EITHER

$\left(\frac{z_1}{z_2}\right)^6 = \frac{2^6 a^6 \mathrm{cis}(0)}{\sqrt{2}^6 \mathrm{cis}\left(\frac{\pi}{2}\right)} \left(= 8a^6\mathrm{cis}\left(-\frac{\pi}{2}\right)\right)$ ***M1 A1 A1***

OR

$\left(\frac{z_1}{z_2}\right)^6 = \left(\frac{2a}{\sqrt{2}}\mathrm{cis}\left(\frac{7\pi}{12}\right)\right)^6$ ***M1 A1***

$= 8a^6\mathrm{cis}\left(-\frac{\pi}{2}\right)$ ***A1***

THEN

$= -8a^6\mathrm{i}$ ***A1***

Note: Accept equivalent angles, in radians or degrees.

Accept alternate answers without cis e.g. $=\frac{8a^6}{\mathrm{i}}$

[7 marks]

Examiners report

Most students had an idea of what to do but were frequently let down in their calculations of the modulus and argument. The most common error was to give the argument of z_2 as $\frac{3\pi}{4}$, failing to recognise that it should be in the fourth quadrant. There were also errors seen in the algebraic manipulation, in particular forgetting to raise the modulus to the power 6.

Let $w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$.

Consider the quadratic equation $z^2 + bz + c = 0$ where $b,\ c \in \mathbb{R},\ z \in \mathbb{C}$. The roots of this equation are α and $\alpha *$ where $\alpha *$ is the complex conjugate of α.

a. Verify that w is a root of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$. [3]

b. (i) Expand $(w-1)(1+w+w^2+w^3+w^4+w^5+w^6)$. [3]

(ii) Hence deduce that $1+w+w^2+w^3+w^4+w^5+w^6 = 0$.

c. Write down the roots of the equation $z^7 - 1 = 0,\ z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram. [3]

d. (i) Given that $\alpha = w + w^2 + w^4$, show that $\alpha * = w^6 + w^5 + w^3$. [10]

(ii) Find the value of b and the value of c.

e. Using the values for b and c obtained in part (d)(ii), find the imaginary part of α, giving your answer in surd form. [4]

Markscheme

a. **EITHER**

$w^7 = \left(\cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}\right)^7$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***A1***

$= 1$ ***A1***

so w is a root ***AG***

OR

$z^7 = 1 = \cos(2\pi k) + \mathrm{i}\sin(2\pi k)$ ***(M1)***

$z = \cos\left(\frac{2\pi k}{7}\right) + \mathrm{i}\sin\left(\frac{2\pi}{7}\right)$ ***A1***

$k = 1 \Rightarrow z = \cos\left(\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2\pi}{7}\right)$ ***A1***

so w is a root ***AG***

[3 marks]

b. (i) $(w-1)(1+w+w^2+w^3+w^4+w^5+w^6)$

$= w+w^2+w^3+w^4+w^5+w^6+w^7-1-w-w^2-w^3-w^4-w^5-w^6$ ***M1***

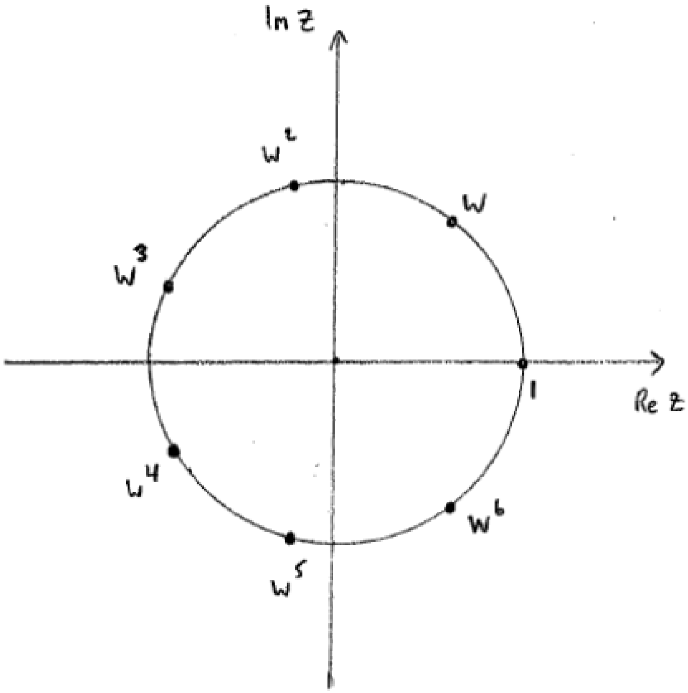
$= w^7 - 1\ (=0)$ ***A1***

(ii) $w^7 - 1 = 0$ and $w - 1 \neq 0$ ***R1***

so $1+w+w^2+w^3+w^4+w^5+w^6 = 0$ ***AG***

[3 marks]

c. the roots are $1,\ w,\ w^2,\ w^3,\ w^4,\ w^5$ and w^6



7 points equidistant from the origin ***A1***

approximately correct angular positions for $1,\ w,\ w^2,\ w^3,\ w^4,\ w^5$ and w^6 ***A1***

Note: Condone use of *cis* notation for the final two ***A*** marks.

Note: For the final ***A*** mark there should be one root in the first quadrant, two in the second, two in the third, one in the fourth, and one on the real axis.

[3 marks]

d. (i) $\alpha* = (w + w^2 + w^4)*$

$= w * + (w^2) * + (w^4)*$ ***A1***

since $= w* = w^6,\ (w^2)* = w^5$ and $(w^4)* = w^3$ ***R1***

$\Rightarrow \alpha* = w^6 + w^5 + w^3$ ***AG***

(ii) $b = -(\alpha + \alpha*)$ (using sum of roots (or otherwise)) ***(M1)***

$b = -(w + w^2 + w^3 + w^4 + w^5 + w^6)$ ***(A1)***

$= -(-1)$

$= 1$ ***A1***

$c = \alpha\alpha*$ (using product of roots (or otherwise)) ***(M1)***

$c = (w + w^2 + w^4)(w^6 + w^5 + w^3)$

EITHER

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4$ ***A1***

$= (w^6 + w^5 + w^4 + w^3 + w^2 + w) + 3$ ***M1***

$= 3 - 1$ ***(A1)***

OR

$= w^{10} + w^9 + w^8 + 3w^7 + w^6 + w^5 + w^4 \left(= w^4(1 + w + w^3)(w^3 + w^2 + 1)\right)$ ***A1***

$= w^4(w^6 + w^5 + w^4 + w^2 + w + 1 + 3w^3)$ ***M1***

$= w^4(w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 + 2w^3)$

$= w^4(2w^3)$ ***(A1)***

THEN

$= 2$ ***A1***

[10 marks]

e. $z^2 + z + 2 = 0 \Rightarrow z = \frac{-1 \pm \mathrm{i}\sqrt{7}}{2}$ ***M1A1***

$\mathrm{Im}(w + w^2 + w^4) > 0$ ***R1***

$\mathrm{Im}\,\alpha = \frac{\sqrt{7}}{2}$ ***A1***

Note: Final ***A*** mark is independent of previous ***R*** mark.

[4 marks]

Examiners report

a. The majority of candidates scored full marks in part (a).

b. The majority of candidates scored full marks in part (b)(i). It was expected to see $w - 1 \neq 0$ stated for the (b)(ii) mark, though some did appreciate this.

c. In part (c), the roots were required to be stated in terms of w. This was sometimes ignored, thankfully not too frequently. Clear Argand diagrams were not often seen, and candidates' general presentation in this area could be improved. Having said this, most scripts were awarded at least 2 of 3 marks available.

d. Part (d) proved to be a good discriminator for the better candidates. The product and sum of roots formulae now seem to be better appreciated, and while only the best scored full marks, a good number were able to demonstrate the result $b = 1$.

e. In part (e), of those candidates who reached this far in the paper, most were able to pick up two or three marks, albeit from sometimes following through incorrect work. A correct reason for choosing $\mathrm{i}\sqrt{7}$ over $-\mathrm{i}\sqrt{7}$ was necessary, but rarely, if ever seen.

Let ω be one of the non-real solutions of the equation $z^3 = 1$.

Consider the complex numbers $p = 1 - 3\mathrm{i}$ and $q = x + (2x + 1)\mathrm{i}$, where $x \in \mathbb{R}$.

a. Determine the value of [4]

(i) $1 + \omega + \omega^2$;

(ii) $1 + \omega^* + (\omega^*)^2$.

b. Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$. [4]

c. Find the values of x that satisfy the equation $|p| = |q|$. [5]

d. Solve the inequality $\mathrm{Re}(pq) + 8 < (\mathrm{Im}(pq))^2$. [6]

Markscheme

a. (i) **METHOD 1**

$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega\neq 1$ ***R1***

METHOD 2

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+(\omega^*)^2=0$ ***A2***

[4 marks]

b. $(\omega-3\omega^2)(\omega^2-3\omega)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1***

EITHER

$=-3\omega^2(\omega^2+\omega+1)+13\omega^3$ ***M1***

$=-3\omega^2\times 0+13\times 1$ ***A1***

OR

$=-3\omega+10-3\omega^2=-3(\omega^2+\omega+1)+13$ ***M1***

$=-3\times 0+13$ ***A1***

OR

substitution by $\omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ in any form ***M1***

numerical values of each term seen ***A1***

THEN

$=13$ ***AG***

[4 marks]

c. $|p|=|q|\Rightarrow\sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)***

$5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)***

$x=1,\ x=-\frac{9}{5}$ ***A1***

[5 marks]

d. $pq=(1-3\mathrm{i})\left(x+(2x+1)\mathrm{i}\right)=(7x+3)+(1-x)\mathrm{i}$ ***M1A1***

$\mathrm{Re}(pq)+8<(\mathrm{Im}(pq))^2\Rightarrow(7x+3)+8<(1-x)^2$ ***M1***

$\Rightarrow x^2-9x-10>0$ ***A1***

$\Rightarrow(x+1)(x-10)>0$ ***M1***

$x<-1,\ x>10$ ***A1***

[6 marks]

Examiners report

a. [N/A]
[N/A]

b. [N/A]
d. [N/A]

---

a. Expand and simplify $\left(x-\frac{2}{x}\right)^4$. [3]

b. Hence determine the constant term in the expansion $(2x^2+1)\left(x-\frac{2}{x}\right)^4$. [2]

Markscheme

a. $\left(x-\frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$ ***(A2)***

Note: Award ***(A1)*** for 3 or 4 correct terms.

Note: Accept combinatorial expressions, *e.g.* $\begin{pmatrix}4\\2\end{pmatrix}$ for 6.

$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$ ***A1***

[3 marks]

b. constant term from expansion of $(2x^2+1)\left(x-\frac{2}{x}\right)^4 = -64+24 = -40$ ***A2***

Note: Award ***A1*** for –64 or 24 seen.

[2 marks]

Examiners report

a. It was disappointing to see many candidates expanding $\left(x-\frac{2}{x}\right)^4$ by first expanding $\left(x-\frac{2}{x}\right)^2$ and then either squaring the result or multiplying twice by $\left(x-\frac{2}{x}\right)$, processes which often resulted in arithmetic errors being made. Candidates at this level are expected to be sufficiently familiar with Pascal’s Triangle to use it in this kind of problem. In (b), some candidates appeared not to understand the phrase ‘constant term’.

b. It was disappointing to see many candidates expanding $\left(x-\frac{2}{x}\right)^4$ by first expanding $\left(x-\frac{2}{x}\right)^2$ and then either squaring the result or multiplying twice by $\left(x-\frac{2}{x}\right)$, processes which often resulted in arithmetic errors being made. Candidates at this level are expected to be sufficiently familiar with Pascal's Triangle to use it in this kind of problem. In (b), some candidates appeared not to understand the phrase "constant term".

---

a. Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}$. [2]

b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in \mathbb{Z}$. [2]

c. Use the principle of mathematical induction to prove that [9]

$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x},\ n \in \mathbb{Z}^+,\ x \neq k\pi$ where $k \in \mathbb{Z}$.

d. Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$. [6]

Markscheme

a. $\sin\frac{\pi}{4}+\sin\frac{3\pi}{4}+\sin\frac{5\pi}{4}+\sin\frac{7\pi}{4}+\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with \) + \) or $-$ signs.

[2 marks]

b. $\frac{1-\cos 2x}{2\sin x}\equiv\frac{1-(1-2\sin^2 x)}{2\sin x}$ ***M1***

$\equiv\frac{2\sin^2 x}{2\sin x}$ ***A1***

$\equiv\sin x$ ***AG***

[2 marks]

c. let $\mathrm{P}(n):\sin x+\sin 3x+\ldots+\sin(2n-1)x\equiv\frac{1-\cos 2nx}{2\sin x}$

if $n=1$

$\mathrm{P}(1):\frac{1-\cos 2x}{2\sin x}\equiv\sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x+\sin 3x+\ldots+\sin(2k-1)x\equiv\frac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words “assume” and “true” appear. Do not award ***M1*** for “let $n=k$” only. Subsequent marks are independent of this ***M1***.

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1):\sin x+\sin 3x+\ldots+\sin(2k-1)x+\sin(2k+1)x\equiv\frac{1-\cos 2(k+1)x}{2\sin x}$

$LHS=\sin x+\sin 3x+\ldots+\sin(2k-1)x+\sin(2k+1)x$ ***M1***

$\equiv\frac{1-\cos 2kx}{2\sin x}+\sin(2k+1)x$ ***A1***

$\equiv\frac{1-\cos 2kx+2\sin x\sin(2k+1)x}{2\sin x}$

$\equiv\frac{1-\cos 2kx+2\sin x\cos x\sin 2kx+2\sin^2 x\cos 2kx}{2\sin x}$ ***M1***

$\equiv\frac{1-\left((1-2\sin^2 x)\cos 2kx-\sin 2x\sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv\frac{1-(\cos 2x\cos 2kx-\sin 2x\sin 2kx)}{2\sin x}$ ***A1***

$\equiv\frac{1-\cos(2kx+2x)}{2\sin x}$ ***A1***

$\equiv\frac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n=k$, then also true for $n=k+1$

as true for $n=1$ then true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

d. **EITHER**

$\sin x + \sin 3x = \cos x \Rightarrow \frac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1 - \cos 4x = 2\sin x \cos x,\ (\sin x \neq 0)$ ***A1***

$\Rightarrow 1 - (1 - 2\sin^2 2x) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1***

$\Rightarrow \sin 2x = 0$ or $\sin 2x = \frac{1}{2}$ ***A1***

$2x = \pi,\ 2x = \frac{\pi}{6}$ and $2x = \frac{5\pi}{6}$

OR

$\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x \cos x = \cos x$ ***M1A1***

$\Rightarrow (2\sin 2x - 1)\cos x = 0,\ (\sin x \neq 0)$ ***M1A1***

$\Rightarrow \sin 2x = \frac{1}{2}$ of $\cos x = 0$ ***A1***

$2x = \frac{\pi}{6},\ 2x = \frac{5\pi}{6}$ and $x = \frac{\pi}{2}$

THEN

$\therefore x = \frac{\pi}{2},\ x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ ***A1***

Note: Do not award the final ***A1*** if extra solutions are seen.

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

a. Darren plays first, find the probability that he wins. [4]

b. The game is now changed so that the ball chosen is replaced after each turn. [3]

Darren still plays first.

Show that the probability of Darren winning has not changed.

Markscheme

a. probability that Darren wins $\mathrm{P}(W) + \mathrm{P}(RRW) + \mathrm{P}(RRRRW)$ ***(M1)***

Note: Only award ***M1*** if three terms are seen or are implied by the following numerical equivalent.

Note: Accept equivalent tree diagram for method mark.

$= \frac{2}{6} + \frac{4}{6} \bullet \frac{3}{5} \bullet \frac{2}{4} + \frac{4}{6} \bullet \frac{3}{5} \bullet \frac{2}{4} \bullet \frac{1}{3} \bullet \frac{2}{2} \quad \left(= \frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right)$ ***A2***

Note: ***A1*** for two correct.

$= \frac{3}{5}$ ***A1***

[4 marks]

b. **METHOD 1**

the probability that Darren wins is given by

$\mathrm{P}(W) + \mathrm{P}(RRW) + \mathrm{P}(RRRRW) + \ldots$ ***(M1)***

Note: Accept equivalent tree diagram with correctly indicated path for method mark.

$\mathrm{P}\ (\text{Darren Win}) = \frac{1}{3} + \frac{2}{3} \bullet \frac{2}{3} \bullet \frac{1}{3} + \frac{2}{3} \bullet \frac{2}{3} \bullet \frac{2}{3} \bullet \frac{2}{3} \bullet \frac{1}{3} + \ldots$

or $= \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \ldots\right)$ ***A1***

$= \frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)$ ***A1***

$= \frac{3}{5}$ ***AG***

METHOD 2

$\mathrm{P}\ (\text{Darren wins}) = \mathrm{P}$

$\mathrm{P} = \frac{1}{3} + \frac{4}{9}\mathrm{P}$ ***M1A2***

$\frac{5}{9}\mathrm{P} = \frac{1}{3}$

$\mathrm{P} = \frac{3}{5}$ ***AG***

[3 marks]

Total [7 marks]

Examiners report

a. [N/A]

b. [N/A]

The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n = 3n^2 - 2n$, where $n \in \mathbb{Z}^+$.

a. Write down the value of u_1. [1]

b. Find the value of u_6. [2]

c. Prove that $\{u_n\}$ is an arithmetic sequence, stating clearly its common difference. [4]

Markscheme

a. $u_1 = 1$ ***A1***

[1 mark]

b. $u_6 = S_6 - S_5 = 31$ ***M1A1***

[2 marks]

c. $u_n = S_n - S_{n-1}$ ***M1***

$= (3n^2 - 2n) - \left(3(n-1)^2 - 2(n-1)\right)$

$= (3n^2 - 2n) - (3n^2 - 6n + 3 - 2n + 2)$

$= 6n - 5$ ***A1***

$d = u_{n+1} - u_n$ ***R1***

$= 6n + 6 - 5 - 6n + 5$

$= (6(n+1) - 5) - (6n - 5)$

$= 6$ (constant) ***A1***

Notes: Award ***R1*** only if candidate provides a clear argument that proves that the difference between **ANY** two consecutive terms of the sequence is constant. Do not accept examples involving particular terms of the sequence nor circular reasoning arguments (*eg* use of formulas of APs to prove that it is an AP). Last ***A1*** is independent of ***R1***.

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

Given that $a_{n-1} = -63$ and $a_n = 16$ find

a. the degree of the polynomial; [4]

b. the value of a_0. [2]

Markscheme

a. the sum of the roots of the polynomial $= \frac{63}{16}$ ***(A1)***

$2\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{63}{16}$ ***M1A1***

Note: The formula for the sum of a geometric sequence must be equated to a value for the ***M1*** to be awarded.

$1-\left(\frac{1}{2}\right)^n = \frac{63}{64} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{1}{64}$

$n = 6$ ***A1***

[4 marks]

b. $\frac{a_0}{a_n} = 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16},\ (\mathrm{a}_n = 16)$ ***M1***

$a_0 = 16 \times 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}$

$a_0 = 2^{-5} \quad \left(= \frac{1}{32}\right)$ ***A1***

[2 marks]

Total [6 marks]

Examiners report

a. [N/A]
b. [N/A]

Solve $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$.

Markscheme

$(\ln x)^2 - (\ln 2)(\ln x) - 2(\ln 2)^2\ (= 0)$

EITHER

$\ln x = \frac{\ln 2 \pm \sqrt{(\ln 2)^2 + 8(\ln 2)^2}}{2}$ ***M1***

$= \frac{\ln 2 \pm 3\ln 2}{2}$ ***A1***

OR

$(\ln x - 2\ln 2)(\ln x + 2\ln 2)\ (= 0)$ ***M1A1***

THEN

$\ln x = 2\ln 2$ or $-\ln 2$ ***A1***

$\Rightarrow x = 4$ or $x = \frac{1}{2}$ ***(M1)A1***

Note: ***(M1)*** is for an appropriate use of a log law in either case, dependent on the previous ***M1*** being awarded, **A1** for both correct answers.

solution is $\frac{1}{2} < x < 4$ ***A1***

[6 marks]

Examiners report

[N/A]

A set of positive integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is used to form a pack of nine cards.

Each card displays one positive integer without repetition from this set. Grace wishes to select four cards at random from this pack of nine cards.

a. Find the number of selections Grace could make if the largest integer drawn among the four cards is either a 5, a 6 or a 7. [3]

b. Find the number of selections Grace could make if at least two of the four integers drawn are even. [4]

Markscheme

a. use of the addition principle with 3 terms ***(M1)***

to obtain ${}^4C_3 + {}^5C_3 + {}^6C_3$ $(= 4 + 10 + 20)$ ***A1***

number of possible selections is 34 ***A1***

[3 marks]

b. **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even) ***(M1)***

$\left({}^5C_2 \times {}^4C_2\right) + \left({}^5C_1 \times {}^4C_3\right) + \left({}^5C_0 \times {}^4C_4\right) \quad (= 60 + 20 + 1)$ ***(M1)A1***

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total ***(M1)***

${}^9C_4 - {}^5C_4 - \left({}^5C_3 \times {}^4C_1\right) \quad (= 126 - 5 - 40)$ ***(M1)A1***

THEN

number of possible selections is 81 ***A1***

[4 marks]

Total [7 marks]

Examiners report

a. As the last question on section A, candidates had to think about the strategy for finding the answers to these two parts. Candidates often had a mark-worthy approach, in terms of considering separate cases, but couldn’t implement it correctly.

b. As the last question on section A, candidates had to think about the strategy for finding the answers to these two parts. Candidates often had a mark-worthy approach, in terms of considering separate cases, but couldn’t implement it correctly.

Solve the equation $\log_2(x+3)+\log_2(x-3)=4$.

Markscheme

$\log_2(x+3)+\log_2(x-3)=4$

$\log_2(x^2-9)=4$ ***(M1)***

$x^2-9=2^4\ (=16)$ ***M1A1***

$x^2=25$

$x=\pm 5$ ***(A1)***

$x=5$ ***A1***

[5 marks]

Examiners report

[N/A]

Find the solution of $\log_2 x-\log_2 5=2+\log_2 3$.

Markscheme

$\log_2 x-\log_2 5=2+\log_2 3$

collecting at least two log terms ***(M1)***

eg $\log_2 \frac{x}{5}=2+\log_2 3$ or $\log_2 \frac{x}{15}=2$

obtaining a correct equation without logs ***(M1)***

eg $\frac{x}{5}=12$ **OR** $\frac{x}{15}=2^2$ ***(A1)***

$x=60$ ***A1***

[4 marks]

Examiners report

[N/A]

Consider the following system of equations:

$$x+y+z=1$$

$$2x+3y+z=3$$

$$x+3y-z=\lambda$$

where $\lambda\in\mathbb{R}$.

a. Show that this system does not have a unique solution for any value of λ . [4]

b. (i) Determine the value of λ for which the system is consistent. [4]

(ii) For this value of λ , find the general solution of the system.

Markscheme

a. using row operations, ***M1***

to obtain 2 equations in the same 2 variables ***A1A1***

for example $y - z = 1$

$2y - 2z = \lambda - 1$

the fact that one of the left hand sides is a multiple of the other left hand side indicates that the equations do not have a unique solution, or equivalent ***R1AG***

[4 marks]

b. (i) $\lambda = 3$ ***A1***

(ii) put $z = \mu$ ***M1***

then $y = 1 + \mu$ ***A1***

and $x = -2\mu$ or equivalent ***A1***

[4 marks]

Examiners report

a. [N/A]

b. [N/A]

---

a. Solve the equation $z^3 = 8\mathrm{i},\ z \in \mathbb{C}$ giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ **and** in the form $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{R}$. [6]

b. Consider the complex numbers $z_1 = 1 + \mathrm{i}$ and $z_2 = 2\left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right)$. [11]

(i) Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

(ii) Calculate $z_1 z_2$ and write in the form $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{R}$.

(iii) Hence find the value of $\tan\frac{5\pi}{12}$ in the form $c + d\sqrt{3}$, where $c,\ d \in \mathbb{Z}$.

(iv) Find the smallest value $p > 0$ such that $(z_2)^p$ is a positive real number.

Markscheme

a. **Note:** Accept answers and working in degrees, throughout.

$z^3 = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(A1)***

attempt the use of De Moivre's Theorem in reverse ***M1***

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right);\ 2\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right);$

$2\left(\cos\left(\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(\frac{9\pi}{6}\right)\right)$ ***A2***

Note: Accept cis form.

$z = \pm\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A1*** for two correct solutions in each of the two lines above.

[6 marks]

b. **Note:** Accept answers and working in degrees, throughout.

(i) $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$ ***A1A1***

(ii) $\left(z_2 = \left(\sqrt{3} + \mathrm{i}\right)\right)$

$z_1 z_2 = (1 + \mathrm{i})\left(\sqrt{3} + \mathrm{i}\right)$ ***M1***

$= \left(\sqrt{3} - 1\right) + \mathrm{i}\left(1 + \sqrt{3}\right)$ ***A1***

(iii) $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\right)$ ***M1A1***

Note: Interpret "hence" as "hence or otherwise".

$\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ ***A1***

$= 2 + \sqrt{3}$ ***M1A1***

Note: Award final ***M1*** for an attempt to rationalise the fraction.

(iv) ${z_2}^p = 2^p\left(\mathrm{cis}\left(\frac{p\pi}{6}\right)\right)$ ***(M1)***

${z_2}^p$ is a positive real number when $p = 12$ ***A1***

Note: Accept a solution based on part (a).

[11 marks]

Total [17 marks]

Examiners report

a. [N/A]

b. [N/A]

a. Let $z = x + \mathrm{i}y$ be any non-zero complex number. [8]

(i) Express $\frac{1}{z}$ in the form $u + \mathrm{i}v$.

(ii) If $z + \frac{1}{z} = k$, $k \in \mathbb{R}$, show that either $y = 0$ or $x^2 + y^2 = 1$.

(iii) Show that if $x^2 + y^2 = 1$ then $|k| \leqslant 2$.

b. Let $w = \cos\theta + \mathrm{i}\sin\theta$. [14]

(i) Show that $w^n + w^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}$.

(ii) Solve the equation $3w^2 - w + 2 - w^{-1} + 3w^{-2} = 0$, giving the roots in the form $x + \mathrm{i}y$.

Markscheme

a. (i) $\frac{1}{z} = \frac{1}{x+\mathrm{i}y} \times \frac{x-\mathrm{i}y}{x-\mathrm{i}y} = \frac{x}{x^2+y^2} - \mathrm{i}\frac{y}{x^2+y^2}$ ***(M1)A1***

(ii) $z + \frac{1}{z} = x + \frac{x}{x^2+y^2} + \mathrm{i}\left(y - \frac{y}{x^2+y^2}\right) = k$ ***(A1)***

for k to be real, $y - \frac{y}{x^2+y^2} = 0 \Rightarrow y(x^2 + y^2 - 1) = 0$ ***M1A1***

hence, $y = 0$ or $x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$ ***AG***

(iii) when $x^2 + y^2 = 1,\ z + \frac{1}{z} = 2x$ ***(M1)A1***

$|x| \leqslant 1$ ***R1***

$\Rightarrow |k| \leqslant 2$ ***AG***

[8 marks]

b. (i) $w^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$ ***M1A1***

$\Rightarrow w^n + w^{-n} = (\cos n\theta + \mathrm{i}\sin n\theta) + (\cos n\theta - \mathrm{i}\sin n\theta) = 2\cos n\theta$ ***M1AG***

(ii) (rearranging)

$3(w^2 + w^{-2}) - (w + w^{-1}) + 2 = 0$ ***(M1)***

$\Rightarrow 3(2\cos 2\theta) - 2\cos\theta + 2 = 0$ ***A1***

$\Rightarrow 2(3\cos 2\theta - \cos\theta + 1) = 0$

$\Rightarrow 3(2\cos^2\theta - 1) - \cos\theta + 1 = 0$ ***M1***

$\Rightarrow 6\cos^2\theta - \cos\theta - 2 = 0$ ***A1***

$\Rightarrow (3\cos\theta - 2)(2\cos\theta + 1) = 0$ ***M1***

$\therefore \cos\theta = \frac{2}{3},\ \cos\theta = -\frac{1}{2}$ ***A1A1***

$\cos\theta = \frac{2}{3} \Rightarrow \sin\theta = \pm\frac{\sqrt{5}}{3}$ ***A1***

$\cos\theta = -\frac{1}{2} \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$ ***A1***

$\therefore w = \frac{2}{3} \pm \frac{\mathrm{i}\sqrt{5}}{3}, -\frac{1}{2} \pm \frac{\mathrm{i}\sqrt{3}}{2}$ ***A1A1***

Note: Allow ***FT*** from incorrect $\cos\theta$ and/or $\sin\theta$.

[14 marks]

Examiners report

a. A large number of candidates did not attempt part (a), or did so unsuccessfully.

b. It was obvious that many candidates had been trained to answer questions of the type in part (b), and hence of those who attempted it, many did so successfully. Quite a few however failed to find all solutions.

Determine the roots of the equation $(z+2\mathrm{i})^3 = 216\mathrm{i}$, $z \in \mathbb{C}$, giving the answers in the form $z = a\sqrt{3} + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$.

Markscheme

METHOD 1

$216\mathrm{i} = 216\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$ ***A1***

$z + 2\mathrm{i} = \sqrt[3]{216}\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) = \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)^{\frac{1}{3}}$ ***(M1)***

$z + 2\mathrm{i} = 6\left(\cos\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right)\right)$ ***A1***

$z_1 + 2\mathrm{i} = 6\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2} + \frac{\mathrm{i}}{2}\right) = 3\sqrt{3} + 3\mathrm{i}$

$z_2 + 2\mathrm{i} = 6\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right) = 6\left(\frac{-\sqrt{3}}{2} + \frac{\mathrm{i}}{2}\right) = -3\sqrt{3} + 3\mathrm{i}$

$z_3 + 2\mathrm{i} = 6\left(\cos\frac{3\pi}{2} + \mathrm{i}\sin\frac{3\pi}{2}\right) = -6\mathrm{i}$ ***A2***

Note: Award ***A1A0*** for one correct root.

so roots are $z_1 = 3\sqrt{3} + \mathrm{i}$, $z_2 = -3\sqrt{3} + \mathrm{i}$ and $z_3 = -8\mathrm{i}$ ***M1A1***

Note: Award ***M1*** for subtracting 2i from their three roots.

METHOD 2

$\left(a\sqrt{3} + (b+2)\mathrm{i}\right)^3 = 216\mathrm{i}$

$\left(a\sqrt{3}\right)^3 + 3\left(a\sqrt{3}\right)^2(b+2)\mathrm{i} - 3\left(a\sqrt{3}\right)(b+2)^2 - \mathrm{i}(b+2)^3 = 216\mathrm{i}$ ***M1A1***

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 + \mathrm{i}\left(3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3\right) = 216\mathrm{i}$

$\left(a\sqrt{3}\right)^3 - 3\left(a\sqrt{3}\right)(b+2)^2 = 0$ and $3\left(a\sqrt{3}\right)^2(b+2) - (b+2)^3 = 216$ ***M1A1***

$a\left(a^2 - (b+2)^2\right) = 0$ and $9a^2(b+2) - (b+2)^3 = 216$

$a = 0$ or $a^2 = (b+2)^2$

if $a = 0, \ -(b+2)^3 = 216 \Rightarrow b+2 = -6$

$\therefore b = -8$ ***A1***

$(a,\ b) = (0,\ -8)$

if $a^2 = (b+2)^2,\ 9(b+2)^2(b+2) - (b+2)^3 = 216$

$8(b+2)^3 = 216$

$(b+2)^3 = 27$

$b+2 = 3$

$b = 1$

$\therefore a^2 = 9 \Rightarrow a = \pm 3$

$\therefore (a,\ b) = (\pm 3,\ 1)$ ***A1A1***

so roots are $z_1 = 3\sqrt{3} + \mathrm{i},\ z_2 = -3\sqrt{3} + \mathrm{i}$ and $z_3 = -8\mathrm{i}$

METHOD 3

$(z+2\mathrm{i})^3 - (-6\mathrm{i})^3 = 0$

attempt to factorise: ***M1***

$((z+2\mathrm{i}) - (-6\mathrm{i}))\left((z+2\mathrm{i})^2 + (z+2\mathrm{i})(-6\mathrm{i}) + (-6\mathrm{i})^2\right) = 0$ ***A1***

$(z+8\mathrm{i})(z^2 - 2\mathrm{i}z - 28) = 0$ ***A1***

$z + 8\mathrm{i} = 0 \Rightarrow z = -8\mathrm{i}$ ***A1***

$z^2 - 2\mathrm{i}z - 28 = 0 \Rightarrow z = \frac{2\mathrm{i} \pm \sqrt{-4-(4\times 1\times -28)}}{2}$ ***M1***

$z = \frac{2\mathrm{i} \pm \sqrt{108}}{2}$

$z = \frac{2\mathrm{i} \pm 6\sqrt{3}}{2}$

$z = \mathrm{i} \pm 3\sqrt{3}$ ***A1A1***

Special Case:

Note: If a candidate recognises that $\sqrt[3]{216\mathrm{i}} = -6\mathrm{i}$ (anywhere seen), and makes no valid progress in finding three roots, award ***A1*** only.

[7 marks]

Examiners report

[N/A]

The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) State de Moivre’s theorem.

(b) Show that $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$.

(c) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form.

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(e) Check that your result in part (d) is true for $\theta = \frac{\pi}{4}$.

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta d\theta$.

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5\theta d\theta$, explaining your reasoning.

Markscheme

(a) any appropriate form, *e.g.* $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta)$ ***A1***

[1 mark]

(b) $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ ***A1***

$\frac{1}{z^n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ ***(M1)***

$= \cos n\theta - \mathrm{i}\sin(n\theta)$ ***A1***

therefore $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$ ***AG***

[3 marks]

(c) $\left(z - \frac{1}{z}\right)^5 = z^5 + \binom{5}{1} z^4\left(-\frac{1}{z}\right) + \binom{5}{2} z^3\left(-\frac{1}{z}\right)^2 + \binom{5}{3} z^2\left(-\frac{1}{z}\right)^3 + \binom{5}{4} z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$ ***(M1)(A1)***

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ ***A1***

[3 marks]

(d) $\left(z - \frac{1}{z}\right)^5 = z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ ***M1A1***

$(2\mathrm{i}\sin\theta)^5 = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$ ***M1A1***

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ ***AG***

[4 marks]

(e) $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\text{LHS} = 16\left(\sin\frac{\pi}{4}\right)^5$

$= 16\left(\frac{\sqrt{2}}{2}\right)^5$

$= 2\sqrt{2} \quad \left(= \frac{4}{\sqrt{2}}\right)$ ***A1***

$\text{RHS} = \sin\left(\frac{5\pi}{4}\right) - 5\sin\left(\frac{3\pi}{4}\right) + 10\sin\left(\frac{\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} - 5\left(\frac{\sqrt{2}}{2}\right) + 10\left(\frac{\sqrt{2}}{2}\right)$ ***M1A1***

Note: Award ***M1*** for attempted substitution.

$= 2\sqrt{2} \quad \left(= \frac{4}{\sqrt{2}}\right)$ ***A1***

hence this is true for $\theta = \frac{\pi}{4}$ ***AG***

[4 marks]

(f) $\int_0^{\frac{\pi}{2}} \sin^5\theta d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta) d\theta$ ***M1***

$= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$ ***A1***

$= \frac{8}{15}$ ***A1***

[4 marks]

(g) $\int_0^{\frac{\pi}{2}} \cos^5\theta \mathrm{d}\theta = \frac{8}{15}$, with appropriate reference to symmetry and graphs. ***A1R1R1***

Note: Award first ***R1*** for partially correct reasoning *e.g.* sketches of graphs of sin and cos.

Award second ***R1*** for fully correct reasoning involving $\sin^5$ and $\cos^5$.

[3 marks]

Total [22 marks]

Examiners report

Many students in b) substituted for the second term (again not making the connection to part a)) on the LHS and multiplied by the conjugate, which some managed well but it is inefficient. The binomial expansion was done well even if students did not do the earlier part. The connection between d) and f) was missed by many which lead to some creative attempts at the integral. Very few attempted the last part and of those many attempted another integral, ignoring the hence, while others related to the graph of sin and cos but not to the particular graphs here.

Three girls and four boys are seated randomly on a straight bench. Find the probability that the girls sit together and the boys sit together.

Markscheme

METHOD 1

total number of arrangements 7! ***(A1)***

number of ways for girls and boys to sit together $= 3! \times 4! \times 2$ ***(M1)(A1)***

Note: Award ***M1A0*** if the 2 is missing.

probability $\frac{3! \times 4! \times 2}{7!}$ ***M1***

Note: Award ***M1*** for attempting to write as a probability.

$\frac{2 \times 3 \times 4! \times 2}{7 \times 6 \times 5 \times 4!}$

$= \frac{2}{35}$ ***A1***

Note: Award ***A0*** if not fully simplified.

METHOD 2

$\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}+\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}$ ***(M1)A1A1***

Note: Accept $\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times 2$ or $\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times 2$.

$=\frac{2}{35}$ ***(M1)A1***

Note: Award ***A0*** if not fully simplified.

[5 marks]

Examiners report

[N/A]

The following system of equations represents three planes in space.

$$x+3y+z=-1$$
$$x+2y-2z=15$$
$$2x+y-z=6$$

Find the coordinates of the point of intersection of the three planes.

Markscheme

EITHER

eliminating a variable, x, for example to obtain $y+3z=-16$ and $-5y-3z=8$ ***M1A1***

attempting to find the value of one variable ***M1***

point of intersection is $(-1,\ 2,\ -6)$ ***A1A1A1***

OR

attempting row reduction of relevant matrix, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & 3 & 1 & -1 \\ 1 & 2 & -2 & 15 \end{array}\right)$ ***M1***

correct matrix with two zeroes in a column, *eg*. $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 0 & 5 & 3 & -8 \\ 0 & 1 & 3 & -16 \end{array}\right)$ ***A1***

further attempt at reduction ***M1***

point of intersection is $(-1,\ 2,\ -6)$ ***A1A1A1***

Note: Allow solution expressed as $x=-1,\ y=2,\ z=-6$ for final ***A*** marks.

[6 marks]

Examiners report

This provided a generally easy start for many candidates. Most successful candidates obtained their answer through row reduction of a suitable matrix. Those choosing an alternative method often made slips in their algebra.

---

a. Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. [2]

b. Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3]

c. Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. [3]

d. **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. [6]

e. By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. [8]

Markscheme

a. $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\,(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$ ***(M1)***

$= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$ ***A1***

[2 marks]

b. from De Moivre's theorem

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ ***(M1)***

$\cos 3\theta + \mathrm{i}\sin 3\theta = \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + \mathrm{i}\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$

equating real parts ***M1***

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right)$ ***A1***

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ ***AG***

Note: Do not award marks if part (a) is not used.

[3 marks]

c. $(\cos\theta + \mathrm{i}\sin\theta)^5 =$

$\cos^5\theta + 5\cos^4\theta\,(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2 + 10\cos^2\theta(\mathrm{i}\sin\theta)^3 + 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$ ***(A1)***

from De Moivre's theorem

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***M1***

$= \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta\left(1 - \cos^2\theta\right)^2$ ***A1***

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ***AG***

Note: If compound angles used in (b) and (c), then marks can be allocated in (c) only.

[3 marks]

d. $\cos 5\theta + \cos 3\theta + \cos \theta$

$= \left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right) + \left(4\cos^3\theta - 3\cos\theta\right) + \cos\theta = 0$ ***M1***

$16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0$ ***A1***

$\cos\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right) = 0$

$\cos\theta\left(4\cos^2\theta - 3\right)\left(4\cos^2\theta - 1\right) = 0$ ***A1***

$\therefore \cos\theta = 0; \pm\frac{\sqrt{3}}{2}; \pm\frac{1}{2}$ ***A1***

$\therefore \theta = \pm\frac{\pi}{6}; \pm\frac{\pi}{3}; \pm\frac{\pi}{2}$ ***A2***

[6 marks]

e. $\cos 5\theta = 0$

$5\theta = \ldots \frac{\pi}{2}; \left(\frac{3\pi}{2}; \frac{5\pi}{2}\right); \frac{7\pi}{2}; \ldots$ ***(M1)***

$\theta = \ldots \frac{\pi}{10}; \left(\frac{3\pi}{10}; \frac{5\pi}{10}\right); \frac{7\pi}{10}; \ldots$ ***(M1)***

Note: These marks can be awarded for verifications later in the question.

now consider $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ ***M1***

$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$

$\cos^2\theta = \frac{20\pm\sqrt{400-4(16)(5)}}{32}; \cos\theta = 0$ ***A1***

$\cos\theta = \pm\sqrt{\frac{20\pm\sqrt{400-4(16)(5)}}{32}}$

$\cos\frac{\pi}{10} = \sqrt{\frac{20+\sqrt{400-4(16)(5)}}{32}}$ since max value of cosine $\Rightarrow$ angle closest to zero ***R1***

$\cos\frac{\pi}{10} = \sqrt{\frac{4.5+4\sqrt{25-4(5)}}{4.8}} = \sqrt{\frac{5+\sqrt{5}}{8}}$ ***A1***

$\cos\frac{7\pi}{10} = -\sqrt{\frac{5-\sqrt{5}}{8}}$ ***A1A1***

[8 marks]

Examiners report

a. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

b. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

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e. This question proved to be very difficult for most candidates. Many had difficulties in following the instructions and attempted to use addition formulae rather than binomial expansions. A small number of candidates used the results given and made a good attempt to part (d) but very few answered part (e).

Given that $y = \frac{1}{1-x}$, use mathematical induction to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{n!}{(1-x)^{n+1}}$, $n \in \mathbb{Z}^+$.

Markscheme

proposition is true for $n = 1$ since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1-x)^2}$ ***M1***

$= \frac{1!}{(1-x)^2}$ ***A1***

Note: Must see the $1!$ for the ***A1***.

assume true for $n = k$, $k \in \mathbb{Z}^+$, *i.e.* $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{k!}{(1-x)^{k+1}}$ ***M1***

consider $\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}$ ***(M1)***

$= (k+1)k!(1-x)^{-(k+1)-1}$ ***A1***

$= \frac{(k+1)!}{(1-x)^{k+2}}$ ***A1***

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers ***R1***

Note: The final ***R1*** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

Examiners report

Most candidates were awarded good marks for this question. A disappointing minority thought that the $(k+1)$th derivative was the (k)th derivative multiplied by the first derivative. Providing an acceptable final statement remains a perennial issue.

Consider the expansion of $(1+x)^n$ in ascending powers of x, where $n \geqslant 3$.

The coefficients of the second, third and fourth terms of the expansion are consecutive terms of an arithmetic sequence.

a. Write down the first four terms of the expansion. [2]

b. (i) Show that $n^3 - 9n^2 + 14n = 0$. [6]

(ii) Hence find the value of n.

Markscheme

a. $1,\ nx,\ \frac{n(n-1)}{2}x^2,\ \frac{n(n-1)(n-2)}{6}x^3$ ***A1A1***

Note: Award ***A1*** for the first two terms and ***A1*** for the next two terms.

Note: Accept $\begin{pmatrix} n \\ r \end{pmatrix}$ notation.

Note: Allow the terms seen in the context of an arithmetic sum.

Note: Allow unsimplified terms, *eg*, those including powers of 1 if seen.

[2 marks]

b. (i) **EITHER**

using $u_3 - u_2 = u_4 - u_3$ ***(M1)***

$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$ ***A1***

attempting to remove denominators and expanding (or vice versa) ***M1***

$3n^2 - 9n = n^3 - 6n^2 + 5n$ (or equivalent, *eg*, $6n^2 - 12n = n^3 - 3n^2 + 2n$) ***A1***

OR

using $u_2 + u_4 = 2u_3$ ***(M1)***

$n + \frac{n(n-1)(n-2)}{6} = n(n-1)$ ***(A1)***

attempting to remove denominators and expanding (or vice versa) ***M1***

$6n + n^3 - 3n^2 + 2n = 6n^2 - 6n$ (or equivalent) ***(A1)***

THEN

$n^3 - 9n^2 + 14n = 0$ ***AG***

(ii) $n(n-2)(n-7) = 0$ or $(n-2)(n-7) = 0$ ***(A1)***

$n = 7$ only (as $n \geqslant 3$) ***A1***

[6 marks]

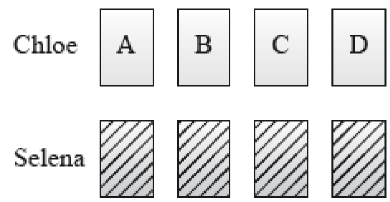
Examiners report

a. This was another question that was very well answered by most candidates.

b. This was another question that was very well answered by most candidates. Some struggled in part (b) by attempting to find an expression for d, a common difference, then substituting this in to further equations, where algebra tended to falter. The most fruitful technique was to apply $u_3 - u_2 = u_4 - u_3$. Good presentation often helped candidates reach the final result. Correct factorisation was more often seen than not in the final section, though a small number thought it judicious to guess the correct answer(s) here.

Chloe and Selena play a game where each have four cards showing capital letters A, B, C and D.

Chloe lays her cards face up on the table in order A, B, C, D as shown in the following diagram.



Selena shuffles her cards and lays them face down on the table. She then turns them over one by one to see if her card matches with Chloe's card directly above.
Chloe wins if **no** matches occur; otherwise Selena wins.

Chloe and Selena repeat their game so that they play a total of 50 times.

Suppose the discrete random variable *X* represents the number of times Chloe wins.

a. Show that the probability that Chloe wins the game is $\frac{3}{8}$. [6]

b.i. Determine the mean of *X*. [3]

b.ii. Determine the variance of *X*. [2]

Markscheme

a. **METHOD 1**

number of possible "deals" $= 4! = 24$ ***A1***

consider ways of achieving "no matches" (Chloe winning):

Selena could deal B, C, D (*ie*, 3 possibilities)

as her first card ***R1***

for each of these matches, there are only 3 possible combinations for the remaining 3 cards ***R1***

so no. ways achieving no matches $= 3 \times 3 = 9$ ***M1A1***

so probability Chloe wins $= \frac{9}{23} = \frac{3}{8}$ ***A1AG***

METHOD 2

number of possible "deals" $= 4! = 24$ ***A1***

consider ways of achieving a match (Selena winning)

Selena card A can match with Chloe card A, giving 6 possibilities for this happening ***R1*** if Selena deals B as her first card, there are only 3 possible combinations for the remaining 3 cards. Similarly for dealing C and dealing D ***R1***

so no. ways achieving one match is $= 6 + 3 + 3 + 3 = 15$ ***M1A1***

so probability Chloe wins $= 1 - \frac{15}{24} = \frac{3}{8}$ ***A1AG***

METHOD 3

systematic attempt to find number of outcomes where Chloe wins (no matches)

(using tree diag. or otherwise) ***M1***

9 found ***A1***

each has probability $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1$ ***M1***

$= \frac{1}{24}$ ***A1***

their 9 multiplied by their $\frac{1}{24}$ ***M1A1***

$= \frac{3}{8}$ ***AG***

[6 marks]

b.i. $X \sim \mathrm{B}\left(50,\ \frac{3}{8}\right)$ ***(M1)***

$\mu = np = 50 \times \frac{3}{8} = \frac{150}{8} \left(= \frac{75}{4}\right) (= 18.75)$ ***(M1)A1***

[3 marks]

b.ii $\sigma^2 = np(1-p) = 50 \times \frac{3}{8} \times \frac{5}{8} = \frac{750}{64} \left(= \frac{375}{32}\right) (= 11.7)$ ***(M1)A1***

[2 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$.

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\mathrm{i},\ a,\ b \in \mathbb{R}$.

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$.

Markscheme

(a) **METHOD 1**

$\frac{1}{2+3\mathrm{i}} + \frac{1}{3+2\mathrm{i}} = \frac{2-3\mathrm{i}}{4+9} + \frac{3-2\mathrm{i}}{9+4}$ ***M1A1***

$\frac{10}{w} = \frac{5-5\mathrm{i}}{13}$ ***A1***

$w = \frac{130}{5-5\mathrm{i}}$

$= \frac{130 \times 5 \times (1+\mathrm{i})}{50}$

$w = 13 + 13\mathrm{i}$ ***A1***

[4 marks]

METHOD 2

$\frac{1}{2+3\text{i}}+\frac{1}{3+2\text{i}}=\frac{3+2\text{i}+2+3\text{i}}{(2+3\text{i})(3+2\text{i})}$ ***M1A1***

$\frac{10}{w}=\frac{5+5\text{i}}{13\text{i}}$ ***A1***

$\frac{w}{10}=\frac{13\text{i}}{5+5\text{i}}$

$w=\frac{130\text{i}}{(5+5\text{i})}\times\frac{(5-5\text{i})}{(5-5\text{i})}$

$=\frac{650+650\text{i}}{50}$

$=13+13\text{i}$ ***A1***

[4 marks]

(b) $w^*=13-13\text{i}$ ***A1***

$z=\sqrt{338}e^{-\frac{\pi}{4}\text{i}}\ \left(=13\sqrt{2}e^{-\frac{\pi}{4}\text{i}}\right)$ ***A1A1***

Note: Accept $\theta=\frac{7\pi}{4}$.

Do not accept answers for θ given in degrees.

[3 marks]

Total [7 marks]

Examiners report

[N/A]

An arithmetic sequence has first term a and common difference d, $d\neq 0$. The 3^{rd}, 4^{th} and 7^{th} terms of the arithmetic sequence are the first three terms of a geometric sequence.

a. Show that $a=-\frac{3}{2}d$. [3]

b. Show that the 4^{th} term of the geometric sequence is the 16^{th} term of the arithmetic sequence. [5]

Markscheme

a. let the first three terms of the geometric sequence be given by u_1 , u_1r , u_1r^2

$\therefore u_1=a+2d$, $u_1r=a+3d$ and $u_1r^2=a+6d$ ***(M1)***

$\frac{a+6d}{a+3d}=\frac{a+3d}{a+2d}$ ***A1***

$a^2+8ad+12d^2=a^2+6ad+9d^2$ ***A1***

$2a+3d=0$

$a=-\frac{3}{2}d$ ***AG***

[3 marks]

b. $u_1=\frac{d}{2}$, $u_1r=\frac{3d}{2}$, $\left(u_1r^2=\frac{9d}{2}\right)$ ***M1***

r = 3 ***A1***

geometric 4^{th} term $u_1r^3=\frac{27d}{2}$ ***A1***

arithmetic 16^{th} term $a+15d=-\frac{3}{2}d+15d$ ***M1***

$= \frac{27d}{2}$ ***A1***

Note: Accept alternative methods.

[3 marks]

Examiners report

a. This question was done well by many students. Those who did not do it well often became involved in convoluted algebraic processes that complicated matters significantly. There were a number of different approaches taken which were valid.

b. This question was done well by many students. Those who did not do it well often became involved in convoluted algebraic processes that complicated matters significantly. There were a number of different approaches taken which were valid.

Use the principle of mathematical induction to prove that

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$, where $n \in \mathbb{Z}^+$.

Markscheme

if $n = 1$

$\text{LHS} = 1\,;\ \text{RHS} = 4 - \frac{3}{2^0} = 4 - 3 = 1$ ***M1***

hence true for $n = 1$

assume true for $n = k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

if $n = k + 1$

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2)-(k+1)}{2^k} = 4 - \frac{k+3}{2^k}\left(= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n = k$ then also true for $n = k + 1$, as true for $n = 1$, so true (for all $n \in \mathbb{Z}^+$) ***R1***

Note: Award the final ***R1*** only if the first four marks have been awarded.

[7 marks]

Examiners report

[N/A]

Expand $(2-3x)^5$ in ascending powers of x, simplifying coefficients.

Markscheme

clear attempt at binomial expansion for exponent 5 ***M1***

$2^5+5\times 2^4\times(-3x)+\frac{5\times 4}{2}\times 2^3\times(-3x)^2+\frac{5\times 4\times 3}{6}\times 2^2\times(-3x)^3$

$+\frac{5\times 4\times 3\times 2}{24}\times 2\times(-3x)^4+(-3x)^5$ ***(A1)***

Note: Only award ***M1*** if binomial coefficients are seen.

$=32-240x+720x^2-1080x^3+810x^4-243x^5$ ***A2***

Note: Award ***A1*** for correct moduli of coefficients and powers. ***A1*** for correct signs.

Total [4 marks]

Examiners report

Generally well done. The majority of candidates obtained a quintic with correct alternating signs. A few candidates made arithmetic errors. A small number of candidates multiplied out the linear expression, often correctly.

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a. The random variable X has the Poisson distribution $\mathrm{Po}(m)$. Given that $\mathrm{P}(X>0)=\frac{3}{4}$, find the value of m in the form $\ln a$ where a is an integer. [3]

b. The random variable Y has the Poisson distribution $\mathrm{Po}(2m)$. Find $\mathrm{P}(Y>1)$ in the form $\frac{b-\ln c}{c}$ where b and c are integers. [4]

Markscheme

a. $\mathrm{P}(X>0)=1-\mathrm{P}(X=0)$ ***(M1)***

$\Rightarrow 1-\mathrm{e}^{-m}=\frac{3}{4}$ or equivalent ***A1***

$\Rightarrow m=\ln 4$ ***A1***

[3 marks]

b. $\mathrm{P}(Y>1)=1-\mathrm{P}(Y=0)-\mathrm{P}(Y=1)$ ***(M1)***

$=1-\mathrm{e}^{-2\ln 4}-\mathrm{e}^{-2\ln 4}\times 2\ln 4$ ***A1***

recognition that $2\ln 4=\ln 16$ ***(A1)***

$\mathrm{P}(Y>1)=\frac{15-\ln 16}{16}$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]

Use mathematical induction to prove that $(2n)! \geq 2^n (n!)^2,\ n \in \mathbb{Z}^+$.

Markscheme

let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2,\ n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1 (1!)^2 = 2$ so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2,\ n \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$(2(k+1))! = (2k+2)(2k+1)(2k)!$ ***M1***

$(2(k+1))! \geq (2k+2)(2k+1)(k!)^2 2^k$ ***A1***

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2 \quad \text{since} \quad 2k+1 > k+1$ ***R1***

$= 2^{k+1}((k+1)!)^2$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[7 marks]

Examiners report

An easy question, but many candidates exhibited discomfort and poor reasoning abilities. The difficulty for most was that the proposition was expressed in terms of an inequality. Hopefully, as most publishers of IB textbooks have realised, inequalities in such questions are within the syllabus.

Prove by mathematical induction that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$, where $n \in \mathbb{Z}, n \geqslant 3$.

Markscheme

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$$

show true for $n=3$ ***(M1)***

$\mathrm{LHS}=\binom{2}{2}=1 \quad \mathrm{RHS}=\binom{3}{3}=1$ ***A1***

hence true for $n=3$

assume true for $n=k$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}=\binom{k}{3}$ ***M1***

consider for $n=k+1$: $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{k-1}{2}+\binom{k}{2}$ ***(M1)***

$=\binom{k}{3}+\binom{k}{2}$ ***A1***

$=\frac{k!}{(k-3)!3!}+\frac{k!}{(k-2)!2!}\ \left(=\frac{k!}{3!}\left[\frac{1}{(k-3)!}+\frac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$=\frac{k!}{3!}\left[\frac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$=\frac{k!}{3!}\left[\frac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$=\frac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$=\binom{k+1}{3}$

Result is true for $k=3$. If result is true for k it is true for $k+1$. Hence result is true for all $k \geqslant 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

[9 marks]

Examiners report

[N/A]

Find the coefficient of x^8 in the expansion of $\left(x^2-\frac{2}{x}\right)^7$.

Markscheme

each term is of the form $\binom{7}{r}\left(x^2\right)^{7-r}\left(\frac{-2}{x}\right)^r$ ***(M1)***

$=\binom{7}{r}x^{14-2r}(-2)^r x^{-r}$

so $14 - 3r = 8$ ***(A1)***

$r = 2$

so require $\binom{7}{2}(x^2)^5\left(\frac{-2}{x}\right)^2$ (or simply $\binom{7}{2}(-2)^2$) ***A1***

$= 21 \times 4$

$= 84$ ***A1***

Note: Candidates who attempt a full expansion, including the correct term, may only be awarded ***M1A0A0A0***.

[4 marks]

Examiners report

[N/A]

Solve the equation $4^{x-1} = 2^x + 8$.

Markscheme

$2^{2x-2} = 2^x + 8$ ***(M1)***

$\frac{1}{4}2^{2x} = 2^x + 8$ ***(A1)***

$2^{2x} - 4 \times 2^x - 32 = 0$ ***A1***

$(2^x - 8)(2^x + 4) = 0$ ***(M1)***

$2^x = 8 \Rightarrow x = 3$ ***A1***

Notes: Do not award final ***A1*** if more than 1 solution is given.

[5 marks]

Examiners report

Very few candidates knew how to solve this equation. A significant number guessed the answer using trial and error after failed attempts to solve it. A number of misconceptions were identified involving properties of logarithms and exponentials.

A geometric sequence $\{u_n\}$, with complex terms, is defined by $u_{n+1} = (1 + \mathrm{i})u_n$ and $u_1 = 3$.

(a) Find the fourth term of the sequence, giving your answer in the form $x + y\mathrm{i},\ x,\ y \in \mathbb{R}$.

(b) Find the sum of the first 20 terms of $\{u_n\}$, giving your answer in the form $a \times (1 + 2^m)$ where $a \in \mathbb{C}$ and $m \in \mathbb{Z}$ are to be determined.

A second sequence $\{v_n\}$ is defined by $v_n = u_n u_{n+k},\ k \in \mathbb{N}$.

(c) (i) Show that $\{v_n\}$ is a geometric sequence.

(ii) State the first term.

(iii) Show that the common ratio is independent of k.

A third sequence $\{w_n\}$ is defined by $w_n = |u_n - u_{n+1}|$.

(d) (i) Show that $\{w_n\}$ is a geometric sequence.

(ii) State the geometrical significance of this result with reference to points on the complex plane.

Markscheme

(a) $r = 1 + \mathrm{i}$ ***(A1)***

$u_4 = 3(1+\mathrm{i})^3$ ***M1***

$= -6 + 6\mathrm{i}$ ***A1***

[3 marks]

(b) $S_{20} = \frac{\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$ ***(M1)***

$= \frac{3\left((2i)^{10} - 1\right)}{\mathrm{i}}$ ***(M1)***

Note: Only one of the two ***M1***s can be implied. Other algebraic methods may be seen.

$= \frac{3(-2^{10}-1)}{\mathrm{i}}$ ***(A1)***

$= 3\mathrm{i}\left(2^{10} + 1\right)$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$ ***M1***

$9(1+\mathrm{i})^k(1+i)^{2n-2}$ ***A1***

$= 9(1+\mathrm{i})^k\left((1+i)^2\right)^{n-1}\left(= 9(1+\mathrm{i})^k(2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence ***R1AG***

Notes: Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further.

If the final expression for v_n is $9(1+\mathrm{i})^k(1+i)^{2n-2}$ award ***M1A1R0***.

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$ ***M1***

$= (1+\mathrm{i})(1+\mathrm{i})$ ***A1***

this is a constant, hence sequence is geometric ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$ ***A1***

(iii) common ratio is $(1+i)^2$ $(= 2i)$ (which is independent of k) ***A1***

[5 marks]

(d) (i) **METHOD 1**

$w_n \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$ ***M1***

$= 3|1+i|^{n-1}|1-(1+\mathrm{i})|$ ***M1***

$= 3|1+i|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

METHOD 2

$w_n = |u_n - (1+\mathrm{i})u_n|$ ***M1***

$= |u_n|\,|-\mathrm{i}|$

$= |u_n|$ ***A1***

$= \left|3(1+\mathrm{i})^{n-1}\right|$

$= 3|(1+\mathrm{i})|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence ***R1***

Note: Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]

Examiners report

[N/A]

On Saturday, Alfred and Beatrice play 6 different games against each other. In each game, one of the two wins. The probability that Alfred wins any one of these games is $\frac{2}{3}$.

a. Show that the probability that Alfred wins exactly 4 of the games is $\frac{80}{243}$. [3]

b. (i) Explain why the total number of possible outcomes for the results of the 6 games is 64. [4]

(ii) By expanding $(1+x)^6$ and choosing a suitable value for *x*, prove

$$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

(iii) State the meaning of this equality in the context of the 6 games played.

c. The following day Alfred and Beatrice play the 6 games again. Assume that the probability that Alfred wins any one of these games is still $\frac{2}{3}$. [9]

(i) Find an expression for the probability Alfred wins 4 games on the first day and 2 on the second day. Give your answer in the form $\binom{6}{r}^2 \left(\frac{2}{3}\right)^s \left(\frac{1}{3}\right)^t$ where the values of *r*, *s* and *t* are to be found.

(ii) Using your answer to (c) (i) and 6 similar expressions write down the probability that Alfred wins a total of 6 games over the two days as the sum of 7 probabilities.

(iii) Hence prove that $\binom{12}{6} = \binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2$.

d. Alfred and Beatrice play n games. Let A denote the number of games Alfred wins. The expected value of A can be written as [6]

$\mathrm{E}(A) = \sum_{r=0}^{n} r \binom{n}{r} \frac{a^r}{b^n}$.

(i) Find the values of a and b.

(ii) By differentiating the expansion of $(1+x)^n$, prove that the expected number of games Alfred wins is $\frac{2n}{3}$.

Markscheme

a. $B\left(6, \frac{2}{3}\right)$ ***(M1)***

$p(4) = \binom{6}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$ ***A1***

$\binom{6}{4} = 15$ ***A1***

$= 15 \times \frac{2^4}{3^6} = \frac{80}{243}$ ***AG***

[3 marks]

b. (i) 2 outcomes for each of the 6 games or $2^6 = 64$ ***R1***

(ii) $(1+x)^6 = \binom{6}{0} + \binom{6}{1} x + \binom{6}{2} x^2 + \binom{6}{3} x^3 + \binom{6}{4} x^4 + \binom{6}{5} x^5 + \binom{6}{6} x^6$ ***A1***

Note: Accept ${}^nC_r$ notation or $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

setting $x = 1$ in both sides of the expression ***R1***

Note: Do not award ***R1*** if the right hand side is not in the correct form.

$64 = \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$ ***AG***

(iii) the total number of outcomes = number of ways Alfred can win no games, plus the number of ways he can win one game *etc.* ***R1***

[4 marks]

c. (i) Let $\mathrm{P}(x,\ y)$ be the probability that Alfred wins x games on the first day and y on the second.

$\mathrm{P}(4, 2) = \binom{6}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^2 \times \binom{6}{2} \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^4$ ***M1A1***

$\binom{6}{2}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6$ or $\binom{6}{4}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6$ ***A1***

$r = 2$ or 4, $s = t = 6$

(ii) P(Total = 6) =

P(0, 6) + P(1, 5) + P(2, 4) + P(3, 3) + P(4, 2) + P(5, 1) + P(6, 0) ***(M1)***

$= \binom{6}{0}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \binom{6}{1}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6 + \ldots + \binom{6}{6}^2 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^6$ ***A2***

$$= \frac{2^6}{3^{12}}\left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2\right)$$

Note: Accept any valid sum of 7 probabilities.

(iii) use of $\binom{6}{i} = \binom{6}{6-i}$ ***(M1)***

(can be used either here or in (c)(ii))

P(wins 6 out of 12) $= \binom{12}{6} \times \left(\frac{2}{3}\right)^6 \times \left(\frac{1}{3}\right)^6 = \frac{2^6}{3^{12}}\binom{12}{6}$ ***A1***

$= \frac{2^6}{3^{12}}\left(\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2\right) = \frac{2^6}{3^{12}}\binom{12}{6}$ ***A1***

therefore $\binom{6}{0}^2 + \binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = \binom{12}{6}$ ***AG***

[9 marks]

d. (i) $\mathrm{E}(A) = \sum_{r=0}^{n} r\binom{n}{r}\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{n-r} = \sum_{r=0}^{n} r\binom{n}{r}\frac{2^r}{3^n}$

($a = 2$, $b = 3$) ***M1A1***

Note: ***M0A0*** for $a = 2$, $b = 3$ without any method.

(ii) $n(1+x)^{n-1} = \sum_{r=1}^{n}\binom{n}{r}rx^{r-1}$ ***A1A1***

(sigma notation not necessary)

(if sigma notation used also allow lower limit to be $r = 0$)

let $x = 2$ ***M1***

$n3^{n-1} = \sum_{r=1}^{n}\binom{n}{r}r2^{r-1}$

multiply by 2 and divide by 3^n ***(M1)***

$\frac{2n}{3} = \sum_{r=1}^{n}\binom{n}{r}r\frac{2^r}{3^n}\left(= \sum_{r=0}^{n}\binom{n}{r}\frac{2^r}{3^n}\right)$ ***AG***

[6 marks]

Examiners report

a. This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(a) Candidates need to be aware how to work out binomial coefficients without a calculator

b. This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(b) (ii) A surprising number of candidates chose to work out the values of all the binomial coefficients (or use Pascal's triangle) to make a total of 64 rather than simply putting 1 into the left hand side of the expression.

c. This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

d. This question linked the binomial distribution with binomial expansion and coefficients and was generally well done.

(d) This was poorly done. Candidates were not able to manipulate expressions given using sigma notation.

The function *f* is defined by $f(x) = \mathrm{e}^x \sin x$.

a. Show that $f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$. [3]

b. Obtain a similar expression for $f^{(4)}(x)$. [4]

c. Suggest an expression for $f^{(2n)}(x)$, $n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction. [8]

Markscheme

a. $f'(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$ ***A1***

$f''(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x + \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$ ***A1***

$= 2\mathrm{e}^x \cos x$ ***A1***

$= 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ ***AG***

[3 marks]

b. $f'''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$f^{(4)}(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) + 2\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right) - 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4\mathrm{e}^x \cos\left(x + \frac{\pi}{2}\right)$ ***A1***

$= 4\mathrm{e}^x \sin(x + \pi)$ ***A1***

[4 marks]

c. the conjecture is that

$f^{(2n)}(x) = 2^n \mathrm{e}^x \sin\left(x + \frac{n\pi}{2}\right)$ ***A1***

for $n = 1$, this formula gives

$f''(x) = 2\mathrm{e}^x \sin\left(x + \frac{\pi}{2}\right)$ which is correct ***A1***

let the result be true for $n = k$, $\left(i.e.\ f^{(2k)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)\right)$ ***M1***

consider $f^{(2k+1)}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***M1***

$f^{(2(k+1))}(x) = 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) + 2^k \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right) - 2^k \mathrm{e}^x \sin\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \cos\left(x + \frac{k\pi}{2}\right)$ ***A1***

$= 2^{k+1} \mathrm{e}^x \sin\left(x + \frac{(k+1)\pi}{2}\right)$ ***A1***

therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for $n = 1$

the result is proved by induction. ***R1***

Note: Award the final ***R1*** only if the two ***M*** marks have been awarded.

[8 marks]

Examiners report

[N/A]

a. [N/A]
b. [N/A]
c. [N/A]

Expand and simplify $\left(\frac{x}{y}-\frac{y}{x}\right)^4$.

Markscheme

$\left(\frac{x}{y}-\frac{y}{x}\right)^4=\left(\frac{x}{y}\right)^4+4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right)+6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2+4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3+\left(-\frac{y}{x}\right)^4$ ***(M1)(A1)***

Note: Award ***M1*** for attempt to expand and ***A1*** for correct unsimplified expansion.

$=\frac{x^4}{y^4}-4\frac{x^2}{y^2}+6-4\frac{y^2}{x^2}+\frac{y^4}{x^4}\quad\left(=\frac{x^8-4x^6y^2+6x^4y^4-4x^2y^6+y^8}{x^4y^4}\right)$ ***A1A1***

Note: Award ***A1*** for powers, ***A1*** for coefficients and signs.

Note: Final two ***A*** marks are independent of first ***A*** mark.

[4 marks]

Examiners report

This was generally very well answered. Those who failed to gain full marks often made minor sign slips. A surprising number obtained the correct simplified expression, but continued to rearrange their expressions, often doing so incorrectly. Fortunately, there were no penalties for doing so.

---

a. If $w = 2 + 2\mathrm{i}$, find the modulus and argument of w. [2]

b. Given $z=\cos\left(\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(\frac{5\pi}{6}\right)$, find in its simplest form w^4z^6. [4]

Markscheme

a. modulus $=\sqrt{8}$ ***A1***

argument $=\frac{\pi}{4}$ (accept 45°) ***A1***

Note: ***A0*** if extra values given.

[2 marks]

b. **METHOD 1**

$w^4z^6-64e^{\pi\mathrm{i}}\times e^{5\pi\mathrm{i}}$ ***(A1)(A1)***

Note: Allow alternative notation.

$=64e^{6\pi\mathrm{i}}$ ***(M1)***

$=64$ ***A1***

METHOD 2

$w^4 = -64$ ***(M1)(A1)***

$z^6 = -1$ ***(A1)***

$w^4 z^6 = 64$ ***A1***

[4 marks]

Examiners report

a. Those who tackled this question were generally very successful. A few, with varying success, tried to work out the powers of the complex numbers by multiplying the Cartesian form rather than using de Moivre's Theorem.

b. Those who tackled this question were generally very successful. A few, with varying success, tried to work out the powers of the complex numbers by multiplying the Cartesian form rather than using de Moivre's Theorem.

Given the complex numbers $z_1 = 1 + 3\mathrm{i}$ and $z_2 = -1 - \mathrm{i}$.

a. Write down the exact values of $|z_1|$ and $\arg(z_2)$. [2]

b. Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. [5]

Markscheme

a. $|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$ ***A1A1***

[2 marks]

b. $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus ***(M1)(A1)***

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root ***M1***

obtain $\alpha = 2$ at minimum ***(A1)***

state $\sqrt{2}$ as final answer ***A1***

[5 marks]

Examiners report

a. Disappointingly, few candidates obtained the correct argument for the second complex number, mechanically using arctan(1) but not thinking about the position of the number in the complex plane.

b. Most candidates obtained the correct quadratic or its square root, but few knew how to set about minimising it.

Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that *w* is a root of the equation $z^5 - 1 = 0$.

(b) Show that $(w-1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(c) **Hence** show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

Markscheme

(a) **EITHER**

$w^5 = \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^5$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***A1***

$= 1$ ***A1***

Hence *w* is a root of $z^5 - 1 = 0$ ***AG***

OR

Solving $z^5 = 1$ ***(M1)***

$z = \cos\frac{2\pi}{5}n + \mathrm{i}\sin\frac{2\pi}{5}n\,,\quad n = 0, 1, 2, 3, 4.$ ***A1***

$n = 1$ gives $\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$ which is *w* ***A1***

[3 marks]

(b) $(w-1)(1 + w + w^2 + w^3 + w^4) = w + w^2 + w^3 + w^4 + w^5 - 1 - w - w^2 - w^3 - w^4$ ***M1***

$= w^5 - 1$ ***A1***

Since $w^5 - 1 = 0$ **and** $w \neq 1$, $w^4 + w^3 + w^2 + w + 1 = 0.$ ***R1***

[3 marks]

(c) $1 + w + w^2 + w^3 + w^4 =$

$1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^2 + \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^3 + \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^4$ ***(M1)***

$= 1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5} + \cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}$ ***M1***

$= 1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - \mathrm{i}\sin\frac{4\pi}{5} + \cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}$ ***M1A1A1***

Note: Award ***M1*** for attempting to replace 6π and 8π by 4π and 2π .

Award ***A1*** for correct cosine terms and ***A1*** for correct sine terms.

$= 1 + 2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5} = 0$ ***A1***

Note: Correct methods involving equating real parts, use of conjugates or reciprocals are also accepted.

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ ***AG***

[6 marks]

Note: Use of cis notation is acceptable throughout this question.

Total [12 marks]

Examiners report

Parts (a) and (b) were generally well done, although very few stated that $w \neq 1$ in (b). Part (c), the last question on the paper was challenging.

Those candidates who gained some credit correctly focussed on the real part of the identity and realise that different cosine were related.

The mean of the first ten terms of an arithmetic sequence is 6. The mean of the first twenty terms of the arithmetic sequence is 16. Find the value of the 15^{th} term of the sequence.

Markscheme

METHOD 1

$5(2a + 9d) = 60$ (or $2a + 9d = 12$)   ***M1A1***

$10(2a + 19d) = 320$ (or $2a + 19d = 32$)   ***A1***

solve simultaneously to obtain   ***M1***

$a = -3,\ d = 2$ ***A1***

the 15^{th} term is $-3 + 14 \times 2 = 25$ ***A1***

Note: ***FT*** the final ***A1*** on the values found in the penultimate line.

METHOD 2

with an AP the mean of an even number of consecutive terms equals the mean of the middle terms   ***(M1)***

$\frac{a_{10}+a_{11}}{2} = 16 \quad (\text{or } a_{10} + a_{11} = 32)$ ***A1***

$\frac{a_5+a_6}{2} = 6 \quad (\text{or } a_5 + a_6 = 12)$ ***A1***

$a_{10} - a_5 + a_{11} - a_6 = 20$ ***M1***

$5d + 5d = 20$

$d = 2$ and $a = -3 \quad (\text{or } a_5 = 5 \text{ or } a_{10} = 15)$ ***A1***

the 15^{th} term is $-3 + 14 \times 2 = 25 \quad (\text{or } 5 + 10 \times 2 = 25 \text{ or } 15 + 5 \times 2 = 25)$ ***A1***

Note: ***FT*** the final ***A1*** on the values found in the penultimate line.

[6 marks]

Examiners report

Many candidates had difficulties with this question with the given information often translated into incorrect equations.

A geometric sequence u_1 , u_2 , u_3 , ... has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

a. Find the common ratio of the geometric sequence. [2]

b. An arithmetic sequence v_1 , v_2 , v_3 , ... is such that $v_2 = u_2$ and $v_4 = u_4$. [5]

Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$.

Markscheme

a. $u_1 = 27$

$\frac{81}{2} = \frac{27}{1-r}$ ***M1***

$r = \frac{1}{3}$ ***A1***

[2 marks]

b. $v_2 = 9$

$v_4 = 1$

$2d = -8 \Rightarrow d = -4$ ***(A1)***

$v_1 = 13$ ***(A1)***

$\frac{N}{2}(2 \times 13 - 4(N-1)) > 0$ (accept equality) ***M1***

$\frac{N}{2}(30 - 4N) > 0$

$N(15 - 2N) > 0$

$N < 7.5$ ***(M1)***

$N = 7$ ***A1***

Note: $13 + 9 + 5 + 1 - 3 - 7 - 11 > 0 \Rightarrow N = 7$ or equivalent receives full marks.

[5 marks]

Examiners report

a. Part (a) was well done by most candidates. However (b) caused difficulty to most candidates. Although a number of different approaches were seen, just a small number of candidates obtained full marks for this question.

b. Part (a) was well done by most candidates. However (b) caused difficulty to most candidates. Although a number of different approaches were seen, just a small number of candidates obtained full marks for this question.

---

a. Using the definition of a derivative as $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$, show that the derivative of $\frac{1}{2x+1}$ is $\frac{-2}{(2x+1)^2}$. [4]

b. Prove by induction that the n^{th} derivative of $(2x+1)^{-1}$ is $(-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$. [9]

Markscheme

a. let $f(x) = \frac{1}{2x+1}$ and using the result $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$f'(x) = \lim_{h \to 0}\left(\frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}\right)$ ***M1A1***

$\Rightarrow f'(x) = \lim_{h \to 0}\left(\frac{[2x+1]-[2(x+h)+1]}{h[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \lim_{h \to 0}\left(\frac{-2}{[2(x+h)+1][2x+1]}\right)$ ***A1***

$\Rightarrow f'(x) = \frac{-2}{(2x+1)^2}$ ***AG***

[4 marks]

b. let $y = \frac{1}{2x+1}$

we want to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^n \frac{2^n n!}{(2x+1)^{n+1}}$

let $n = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (-1)^1 \frac{2^1 1!}{(2x+1)^{1+1}}$ ***M1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2}{(2x+1)^2}$ which is the same result as part (a)

hence the result is true for $n = 1$ ***R1***

assume the result is true for $n = k$: $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^k \frac{2^k k!}{(2x+1)^{k+1}}$ ***M1***

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k \frac{2^k k!}{(2x+1)^{k+1}}\right]$ ***M1***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k 2^k k!(2x+1)^{-k-1}\right]$ ***(A1)***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^k 2^k k!(-k-1)(2x+1)^{-k-2} \times 2$ ***A1***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^{k+1} 2^{k+1}(k+1)!(2x+1)^{-k-2}$ ***(A1)***

$\Rightarrow \frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^{k+1} \frac{2^{k+1}(k+1)!}{(2x+1)^{k+2}}$ ***A1***

hence if the result is true for $n = k$, it is true for $n = k + 1$
since the result is true for $n = 1$, the result is proved by mathematical induction ***R1***
Note: Only award final ***R1*** if all the ***M*** marks have been gained.

[9 marks]

Examiners report

a. Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write ‘Let $n = k$’ which is of course meaningless. The conclusion is often of the form ‘True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction’. Credit is only given for a conclusion which includes a statement such as ‘True for $n = k \Rightarrow$ true for $n = k + 1$’.

b. Even though the definition of the derivative was given in the question, solutions to (a) were often disappointing with algebraic errors fairly common, usually due to brackets being omitted or manipulated incorrectly. Solutions to the proof by induction in (b) were often poor. Many candidates fail to understand that they have to assume that the result is true for $n = k$ and then show that this leads to it being true for $n = k + 1$. Many candidates just write ‘Let $n = k$’ which is of course meaningless. The conclusion is often of the form ‘True for $n = 1,\ n = k$ and $n = k + 1$ therefore true by induction’. Credit is only given for a conclusion which includes a statement such as ‘True for $n = k \Rightarrow$ true for $n = k + 1$’.

Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

Markscheme

$n = 1: \ 1^3 + 11 = 12$

$= 3 \times 4$ **or** a multiple of 3   ***A1***
assume the proposition is true for $n = k \quad (ie \ k^3 + 11k = 3 \mathrm{m})$ ***M1***

Note: Do not award ***M1*** for statements with "Let $n = k$".

consider $n = k + 1: \quad (k+1)^3 + 11(k+1)$ ***M1***
$= k^3 + 3k^2 + 3k + 1 + 11k + 11$ ***A1***
$= k^3 + 11k + (3k^2 + 3k + 12)$ ***M1***
$= 3(m + k^2 + k + 4)$ ***A1***

Note: Accept $k^3 + 11k + 3(k^2 + k + 4)$ or statement that $k^3 + 11k + (3k^2 + 3k + 12)$ is a multiple of 3.

true for $n = 1$, and $n = k$ true $\Rightarrow n = k + 1$ true
hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

[7 marks]

Examiners report

It was pleasing to see a great many clear and comprehensive answers for this relatively straightforward induction question. The inductive step only seemed to pose problems for the very weakest candidates. As in previous sessions, marks were mainly lost by candidates writing variations on 'Let $n = k$', rather than 'Assume true for $n = k$'. The final reasoning step still needs attention, with variations on '$n = k + 1$ true $\Rightarrow n = k$ true' evident, suggesting that mathematical induction as a technique is not clearly understood.

The sum, S_n, of the first *n* terms of a geometric sequence, whose n^{th} term is u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n}, \text{ where } a > 0.$$

(a) Find an expression for u_n.

(b) Find the first term and common ratio of the sequence.

(c) Consider the sum to infinity of the sequence.

(i) Determine the values of *a* such that the sum to infinity exists.

(ii) Find the sum to infinity when it exists.

Markscheme

METHOD 1

(a) $u_n = S_n - S_{n-1}$ ***(M1)***

$= \frac{7^n - a^n}{7^n} - \frac{7^{n-1} - a^{n-1}}{7^{n-1}}$ ***A1***

(b) **EITHER**

$u_1 = 1 - \frac{a}{7}$ ***A1***

$u_2 = 1 - \frac{a^2}{7^2} - \left(1 - \frac{a}{7}\right)$ ***M1***

$= \frac{a}{7}\left(1 - \frac{a}{7}\right)$ ***A1***

common ratio $= \frac{a}{7}$ ***A1***

OR

$u_n = 1 - \left(\frac{a}{7}\right)^n - 1 + \left(\frac{a}{7}\right)^{n-1}$ ***M1***

$= \left(\frac{a}{7}\right)^{n-1}\left(1 - \frac{a}{7}\right)$

$= \frac{7-a}{7}\left(\frac{a}{7}\right)^{n-1}$ ***A1***

$u_1 = \frac{7-a}{7}$, common ratio $= \frac{a}{7}$ ***A1A1***

(c) (i) $0 < a < 7$ $\quad$ (accep $a < 7$) ***A1***

(ii) $\;1$ ***A1***

[8 marks]

METHOD 2

(a) $u_n = br^{n-1} = \left(\frac{7-a}{7}\right)\left(\frac{a}{7}\right)^{n-1}$ ***A1A1***

(b) for a GP with first term b and common ratio r

$S_n = \frac{b(1-r^n)}{1-r} = \left(\frac{b}{1-r}\right) - \left(\frac{b}{1-r}\right)r^n$ ***M1***

as $S_n = \frac{7^n - a^n}{7^n} = 1 - \left(\frac{a}{7}\right)^n$

comparing both expressions ***M1***

$\frac{b}{1-r} = 1$ and $r = \frac{a}{7}$

$b = 1 - \frac{a}{7} = \frac{7-a}{7}$

$u_1 = b = \frac{7-a}{7}$, common ratio $= r = \frac{a}{7}$ ***A1A1***

Note: Award method marks if the expressions for b and r are deduced in part (a).

(c) (i) $0 < a < 7$ $\quad$ (accept $a < 7$) ***A1***

(ii) $\;1$ ***A1***

[8 marks]

Examiners report

Many candidates found this question difficult. In (a), few seemed to realise that $u_n = S_n - S_{n-1}$. In (b), few candidates realised that $u_1 = S_1$ and in (c) that S_n could be written as $1 - \left(\frac{a}{7}\right)^n$ from which it follows immediately that the sum to infinity exists when a < 7 and is equal to 1.

Let $y(x) = xe^{3x},\ x \in \mathbb{R}$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. [7]

c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. [5]

Justify whether any such point is a maximum or a minimum.

d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

e. Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = (\mathrm{e}^{3x} + 3x\mathrm{e}^{3x})$ ***M1A1***

[2 marks]

b. let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as $\mathrm{LHS} = \mathrm{RHS}$, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true $(\forall n \in \mathbb{Z}^+)$ ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

c. $\mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1*** point is $\left(-\frac{1}{3},\ -\frac{1}{3\mathrm{e}}\right)$ ***A1***

EITHER

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$

when $x = -\frac{1}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-ve \quad 0 \quad +ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

d. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$ ***A1***

$2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3},\ -\frac{2}{3\mathrm{e}^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	$-ve \quad 0 \quad +ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

Note: Allow 3^{rd} derivative is not zero at $-\frac{2}{3}$

[5 marks]

e.

(general shape including asymptote and through origin) ***A1***

showing minimum and point of inflection ***A1***

Note: Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

Examiners report

a. Well done.

b. The logic of an induction proof was not known well enough. Many candidates used what they had to prove rather than differentiating what they had assumed. They did not have enough experience in doing Induction proofs.

c. Good, some forgot to test for min/max, some forgot to give the y value.

d. Again quite good, some forgot to check for change in curvature and some forgot the y value.

e. Some accurate sketches, some had all the information from earlier parts but could not apply it. The asymptote was often missed.

The first three terms of a geometric sequence are $\sin x,\ \sin 2x$ and $4\sin x\cos^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) Find the common ratio r.

(b) Find the set of values of x for which the geometric series $\sin x + \sin 2x + 4\sin x\cos^2 x + \ldots$ converges.

Consider $x = \arccos\left(\frac{1}{4}\right),\ x > 0$.

(c) Show that the sum to infinity of this series is $\frac{\sqrt{15}}{2}$.

Markscheme

(a) $\sin x,\ \sin 2x$ and $4\sin x\cos^2 x$

$r = \frac{2\sin x\cos x}{\sin x} = 2\cos x$ **A1**

Note: Accept $\frac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \frac{1}{2}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \frac{\sin x}{1 - 2\cos x}$ ***M1***

$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$

$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{\sqrt{15}}{2}$ ***AG***

[3 marks]

Total [7 marks]

Examiners report

[N/A]

Consider $a = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{31} 32$. Given that $a \in \mathbb{Z}$, find the value of a.

Markscheme

$\frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \ldots \times \frac{\log 32}{\log 31}$ ***M1A1***

$= \frac{\log 32}{\log 2}$ ***A1***

$= \frac{5 \log 2}{\log 2}$ ***(M1)***

$= 5$ ***A1***

hence $a = 5$

Note: Accept the above if done in a specific base *eg* $\log_2 x$.

[5 marks]

Examiners report

[N/A]

Find the term independent of x in the binomial expansion of $\left(2x^2 + \frac{1}{2x^3}\right)^{10}$.

Markscheme

attempt at binomial expansion, relevant row of Pascal's triangle or use of general term with binomial coefficient must be seen ***(M1)***

term independent of x is $\binom{10}{4}(2x^2)^6\left(\frac{1}{2x^3}\right)^4$ (or equivalent) ***(A1)(A1)(A1)***

Notes: x's may be omitted. Also accept $\binom{10}{6}$ or 210.

$= 840$ ***A1***

[5 marks]

Examiners report

[N/A]

Given that z is the complex number $x + \mathrm{i}y$ and that $|z| + z = 6 - 2\mathrm{i}$, find the value of x and the value of y.

Markscheme

$\sqrt{x^2+y^2}+x+y\mathrm{i}=6-2\mathrm{i}$ ***(A1)***

equating real and imaginary parts ***M1***

$y=-2$ ***A1***

$\sqrt{x^2+4}+x=6$ ***A1***

$x^2+4=(6-x)^2$ ***M1***

$-32=-12x\Rightarrow x=\frac{8}{3}$ ***A1***

[6 marks]

Examiners report

There were some good solutions to this question, but those who failed to complete the question failed at a variety of different points. Many did not know the definition of the modulus of a complex number and so could not get started at all. Many then did not think to equate real and imaginary parts, and then many failed to solve the resulting irrational equation to be able to find *x*.

Consider a function f, defined by $f(x)=\frac{x}{2-x}$ for $0\leqslant x\leqslant 1$.

a. Find an expression for $(f\circ f)(x)$. [3]

b. Let $F_n(x)=\frac{x}{2^n-(2^n-1)x}$, where $0\leqslant x\leqslant 1$. [8]
Use mathematical induction to show that for any $n\in\mathbb{Z}^+$

$$\underbrace{(f\circ f\circ\ldots\circ f)}_{n\text{ times}}(x)=F_n(x)$$

.

c. Show that $F_{-n}(x)$ is an expression for the inverse of F_n . [6]

d. (i) State $F_n(0)$ and $F_n(1)$. [6]
(ii) Show that $F_n(x)<x$, given $0<x<1$, $n\in\mathbb{Z}^+$.
(iii) For $n\in\mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1} , the *x*-axis and the line $x=1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n .

Markscheme

a. $(f\circ f)(x)=f\left(\frac{x}{2-x}\right)=\frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ ***M1A1***
$(f\circ f)(x)=\frac{x}{4-3x}$ ***A1***
[3 marks]

b. $P(n):\underbrace{(f\circ f\circ\ldots\circ f)}_{n\text{ times}}(x)=F_n(x)$

$P(1):\ f(x)=F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1-(2^1-1)x} = \frac{x}{2-x}$ ***A1A1***

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ ***M1***

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}} \right)(x) = f\left(F_k(x)\right)$ ***(M1)***

$= f\left(\frac{x}{2^k-(2^k-1)x}\right) = \frac{\frac{x}{2^k-(2^k-1)x}}{2-\frac{x}{2^k-(2^k-1)x}}$ ***A1***

$= \frac{x}{2\left(2^k-(2^k-1)x\right)-x} = \frac{x}{2^{k+1}-(2^{k+1}-2)x-x}$ ***A1***

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}} \right)(x) = F_k(f(x))$ ***(M1)***

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k-(2^k-1)\frac{x}{2-x}}$ ***A1***

$= \frac{x}{2^{k+1}-2^k x-2^k x+x}$ ***A1***

THEN

$= \frac{x}{2^{k+1}-(2^{k+1}-1)x} = F_{k+1}(x)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ ***R1***

[8 marks]

c. **METHOD 1**

$x = \frac{y}{2^n-(2^n-1)y} \Rightarrow 2^n x - (2^n-1)xy = y$ ***M1A1***

$\Rightarrow 2^n x = \left((2^n-1)x+1\right)y \Rightarrow y = \frac{2^n x}{(2^n-1)x+1}$ ***A1***

$F_n^{-1}(x) = \frac{2^n x}{(2^n-1)x+1}$ ***A1***

$F_n^{-1}(x) = \frac{x}{\frac{2^n-1}{2^n}x+\frac{1}{2^n}}$ ***M1***

$F_n^{-1}(x) = \frac{x}{(1-2^{-n})x+2^{-n}}$ ***A1***

$F_n^{-1}(x) = \frac{x}{2^{-n}-(2^{-n}-1)x}$ ***AG***

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$ ***M1***

$= F_{-n}\left(\frac{x}{2^n-(2^n-1)x}\right) = \frac{\frac{x}{2^n-(2^n-1)x}}{2^{-n}-(2^{-n}-1)\frac{x}{2^n-(2^n-1)x}}$ ***A1A1***

$= \frac{x}{2^{-n}(2^n-(2^n-1)x)-(2^{-n}-1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$ **M1**

$= F_n\left(\frac{x}{2^{-n}-(2^{-n}-1)x}\right) = \frac{\frac{x}{2^{-n}-(2^{-n}-1)x}}{2^n-(2^n-1)\frac{x}{2^{-n}-(2^{-n}-1)x}}$ ***A1A1***

$= \frac{x}{2^n(2^{-n}-(2^{-n}-1)x)-(2^n-1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***

[6 marks]

d. (i) $F_n(0) = 0,\ F_n(1) = 1$ ***A1***

(ii) **METHOD 1**

$2^n - (2^n - 1)x - 1 = (2^n - 1)(1 - x)$ ***(M1)***

> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***A1***

so $2^n - (2^n - 1)x > 1$ and $F_n(x) = \frac{x}{2^n - (2^n - 1)x} < \frac{x}{1} (< x)$ ***R1***

$F_n(x) = \frac{x}{2^n - (2^n - 1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***AG***

METHOD 2

$\frac{x}{2^n - (2^n - 1)x} < x \Leftrightarrow 2^n - (2^n - 1)x > 1$ ***(M1)***

$\Leftrightarrow (2^n - 1)x < 2^n - 1$ ***A1***

$\Leftrightarrow x < \frac{2^n - 1}{2^n - 1} = 1$ true in the interval $]0,\ 1[$ ***R1***

(iii) $B_n = 2\left(A_n - \frac{1}{2}\right)\ (= 2A_n - 1)$ ***(M1)A1***

[6 marks]

Examiners report

a. Part a) proved to be an easy 3 marks for most candidates.

b. Part b) was often answered well, and candidates were well prepared in this session for this type of question. Candidates still need to take care when showing explicitly that P(1) is true, and some are still writing 'Let $n = k$' which gains no marks. The inductive step was often well argued, and given in clear detail, though the final inductive reasoning step was incorrect, or appeared rushed, even from the better candidates. 'True for $n = 1$, $n = k$ and $n = k + 1$' is still disappointingly seen, as were some even more unconvincing variations.

c. Part c) was again very well answered by the majority. A few weaker candidates attempted to find an inverse for the individual case n = 1 , but gained no credit for this.

d. Part d) was not at all well understood, with virtually no candidates able to tie together the hints given by connecting the different parts of the question. Rash, and often thoughtless attempts were made at each part, though by this stage some seemed to be struggling through lack of time. The inequality part of the question tended to be 'fudged', with arguments seen by examiners being largely unconvincing and lacking clarity. A tiny number of candidates provided the correct answer to the final part, though a surprising number persisted with what should have been recognised as fruitless working – usually in the form of long-winded integration attempts.

The common ratio of the terms in a geometric series is 2^x .

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40.

Markscheme

(a) $0 < 2^x < 1$ ***(M1)***

$x < 0$ ***A1*** ***N2***

(b) $\frac{35}{1-r} = 40$ ***M1***

$\Rightarrow 40 - 40 \times r = 35$

$\Rightarrow -40 \times r = -5$ ***(A1)***

$\Rightarrow r = 2^x = \frac{1}{8}$ ***A1***

$\Rightarrow x = \log_2 \frac{1}{8}\ (= -3)$ ***A1***

Note: The substitution $r = 2^x$ may be seen at any stage in the solution.

[6 marks]

Examiners report

Part (a) was the first question that a significant majority of candidates struggled with. Only the best candidates were able to find the required set of values. However, it was pleasing to see that the majority of candidates made a meaningful start to part (b). Many candidates gained wholly correct answers to part (b).

Part A If z is a non-zero complex number, we define $L(z)$ by the equation [9]

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \leqslant \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1-\mathrm{i})$;

(iii) $L(-1+\mathrm{i})$.

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2 .

Part B Let f be a function with domain $\mathbb{R}$ that satisfies the conditions, [14]

$f(x+y) = f(x)f(y)$, for all x and y and $f(0) \neq 0$.

(a) Show that $f(0) = 1$.

(b) Prove that $f(x) \neq 0$, for all $x \in \mathbb{R}$.

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation $f'(x) = k\, f(x)$, where $k = f'(0)$.

(d) Solve the differential equation to find an expression for $f(x)$.

Markscheme

Part A. (a) $|z| = z$, $\arg(z) = 0$ ***A1A1***

so $L(z) = \ln z$ ***AG N0***

[2 marks]

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$ ***A1A1 N2***

(ii) $L(1 - \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{7\pi}{4}$ ***A1A1 N2***

(iii) $L(-1 + \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{3\pi}{4}$ ***A1 N1***

[5 marks]

(c) for comparing the product of two of the above results with the third ***M1***

for stating the result $-1 + \mathrm{i} = -1 \times (1 - \mathrm{i})$ and $L(-1 + \mathrm{i}) \neq L(-1) + L(1 - \mathrm{i})$ ***R1***

hence, the property $L(z_1 z_2) = L(z_1) + L(z_2)$

does not hold for all values of z_1 and z_2 ***AG N0***

[2 marks]

Total [9 marks]

Part B. (a) from $f(x + y) = f(x)f(y)$

for $x = y = 0$ ***M1***

we have $f(0 + 0) = f(0)f(0) \Leftrightarrow f(0) = (f(0))^2$ ***A1***

as $f(0) \neq 0$, this implies that $f(0) = 1$ ***R1AG N0***

[3 marks]

(b) **METHOD 1**

from $f(x + y) = f(x)f(y)$

for $y = -x$, we have $f(x - x) = f(x)f(-x) \Leftrightarrow f(0) = f(x)f(-x)$ ***M1A1***

as $f(0) \neq 0$ this implies that $f(x) \neq 0$ ***R1AG N0***

METHOD 2

suppose that, for a value of x, $f(x) = 0$ ***M1***

from $f(x + y) = f(x)f(y)$

for $y = -x$, we have $f(x - x) = f(x)f(-x) \Leftrightarrow f(0) = f(x)f(-x)$ ***A1***

substituting $f(x)$ by 0 gives $f(0) = 0$ which contradicts part (a) ***R1***

therefore $f(x) \neq 0$ for all x. ***AG N0***

[3 marks]

(c) by the definition of derivative

$f'(x) = \lim\limits_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$ ***(M1)***

$= \lim\limits_{h \to 0}\left(\frac{f(x)f(h) - f(x)f(0)}{h}\right)$ ***A1(A1)***

$= \lim\limits_{h \to 0}\left(\frac{f(h) - f(0)}{h}\right)f(x)$ ***A1***

$= f'(0)f(x) \quad (= k\, f(x))$ ***AG*** ***N0***

[4 marks]

(d) $\int \frac{f'(x)}{f(x)} \mathrm{d}x = \int k \mathrm{d}x \Rightarrow \ln f(x) = kx + C$ ***M1A1***

$\ln f(0) = C \Rightarrow C = 0$ ***A1***

$f(x) = \mathrm{e}^{kx}$ ***A1*** ***N1***

Note: Award ***M1A0A0A0*** if no arbitrary constant *C* .

[4 marks]

Total [14 marks]

Examiners report

Part A: Part A was answered well by a fair amount of candidates, with some making mistakes in calculating the arguments of complex numbers, as well as careless mistakes in finding the products of complex numbers.

Part B: Part B proved demanding for most candidates, particularly parts (c) and (d). A surprising number of candidates did not seem to know what was meant by the 'definition of derivative' in part (c) as they attempted to use quotient rule rather than first principles.

The first terms of an arithmetic sequence are $\frac{1}{\log_2 x}, \frac{1}{\log_8 x}, \frac{1}{\log_{32} x}, \frac{1}{\log_{128} x}, \ldots$

Find *x* if the sum of the first 20 terms of the sequence is equal to 100.

Markscheme

METHOD 1

$d = \frac{1}{\log_8 x} - \frac{1}{\log_2 x}$ ***(M1)***

$= \frac{\log_2 8}{\log_2 x} - \frac{1}{\log_2 x}$ ***(M1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$= \frac{2}{\log_2 x}$ ***(A1)***

$\frac{20}{2}\left(2 \times \frac{1}{\log_2 x} + 19 \times \frac{2}{\log_2 x}\right)$ ***M1***

$= \frac{400}{\log_2 x}$ ***(A1)***

$100 = \frac{400}{\log_2 x}$

$\log_2 x = 4 \Rightarrow x = 2^4 = 16$ ***A1***

METHOD 2

20^{th} term $= \frac{1}{\log_{2^{39}} x}$ ***A1***

$100 = \frac{20}{2}\left(\frac{1}{\log_2 x} + \frac{1}{\log_{2^{39}} x}\right)$ ***M1***

$100 = \frac{20}{2}\left(\frac{1}{\log_2 x} + \frac{\log_2 2^{39}}{\log_2 x}\right)$ ***M1(A1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$100 = \frac{400}{\log_2 x}$ ***(A1)***

$\log_2 x = 4 \Rightarrow x = 2^4 = 16$ ***A1***

METHOD 3

$\frac{1}{\log_2 x} + \frac{1}{\log_8 x} + \frac{1}{\log_{32} x} + \frac{1}{\log_{128} x} + \ldots$

$\frac{1}{\log_2 x} + \frac{\log_2 8}{\log_2 x} + \frac{\log_2 32}{\log_2 x} + \frac{\log_2 128}{\log_2 x} + \ldots$ ***(M1)(A1)***

Note: Award this ***M1*** for a correct change of base anywhere in the question.

$= \frac{1}{\log_2 x}(1 + 3 + 5 + \ldots)$ ***A1***

$= \frac{1}{\log_2 x}\left(\frac{20}{2}(2 + 38)\right)$ ***(M1)(A1)***

$100 = \frac{400}{\log_2 x}$

$\log_2 x = 4 \Rightarrow x = 2^4 = 16$ ***A1***

[6 marks]

Examiners report

There were plenty of good answers to this question. Those who realised they needed to make each log have the same base (and a great variety of bases were chosen) managed the question successfully.

Solve the equation $8^{x-1} = 6^{3x}$. Express your answer in terms of $\ln 2$ and $\ln 3$.

Markscheme

METHOD 1

$2^{3(x-1)} = (2 \times 3)^{3x}$ ***M1***

Note: Award ***M1*** for writing in terms of 2 and 3.

$2^{3x} \times 2^{-3} = 2^{3x} \times 3^{3x}$

$2^{-3} = 3^{3x}$ ***A1***

$\ln\left(2^{-3}\right) = \ln\left(3^{3x}\right)$ ***(M1)***

$-3 \ln 2 = 3x \ln 3$ ***A1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

METHOD 2

$\ln 8^{x-1} = \ln 6^{3x}$ ***(M1)***

$(x-1) \ln 2^3 = 3x \ln(2 \times 3)$ ***M1A1***

$3x\ln 2 - 3\ln 2 = 3x\ln 2 + 3x\ln 3$ ***A1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

METHOD 3

$\ln 8^{x-1} = \ln 6^{3x}$ ***(M1)***

$(x-1)\ln 8 = 3x\ln 6$ ***A1***

$x = \frac{\ln 8}{\ln 8 - 3\ln 6}$ ***A1***

$x = \frac{3\ln 2}{\ln\left(\frac{2^3}{6^3}\right)}$ ***M1***

$x = -\frac{\ln 2}{\ln 3}$ ***A1***

[5 marks]

Examiners report

[N/A]

Consider $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

These four points form the vertices of a quadrilateral, *Q*.

a.i. Express *w*^{2} and *w*^{3} in modulus-argument form. [3]

a.ii. Sketch on an Argand diagram the points represented by *w*^{0} , *w*^{1} , *w*^{2} and *w*^{3}. [2]

b. Show that the area of the quadrilateral *Q* is $\frac{21\sqrt{3}}{2}$. [3]

c. Let $z = 2\left(\cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}\right),\ n \in \mathbb{Z}^+$. The points represented on an Argand diagram by $z^0,\ z^1,\ z^2,\ \ldots,\ z^n$ form the vertices of a polygon P_n. [6]

Show that the area of the polygon P_n can be expressed in the form $a\left(b^n - 1\right)\sin\frac{\pi}{n}$, where $a,\ b \in \mathbb{R}$.

Markscheme

a.i. $w^2 = 4\mathrm{cis}\left(\frac{2\pi}{3}\right)$; $w^3 = 8\mathrm{cis}\left(\pi\right)$ ***(M1)A1A1***

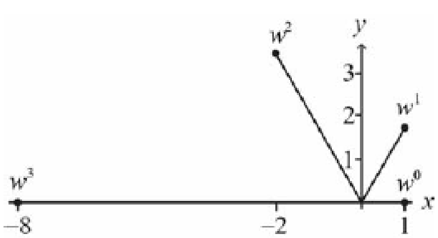
Note: Accept Euler form.

Note: ***M1*** can be awarded for either both correct moduli or both correct arguments.

Note: Allow multiplication of correct Cartesian form for ***M1***, final answers must be in modulus-argument form.

[3 marks]

a.ii.



A1A1

[2 marks]

b. use of area = $\frac{1}{2}ab\ \sin C$ ***M1***

$\frac{1}{2} \times 1 \times 2 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 2 \times 4 \times \sin\frac{\pi}{3} + \frac{1}{2} \times 4 \times 8 \times \sin\frac{\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $C = \frac{\pi}{3}$, ***A1*** for correct moduli.

$= \frac{21\sqrt{3}}{2}$ ***AG***

Note: Other methods of splitting the area may receive full marks.

[3 marks]

c. $\frac{1}{2} \times 2^0 \times 2^1 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^1 \times 2^2 \times \sin\frac{\pi}{n} + \frac{1}{2} \times 2^2 \times 2^3 \times \sin\frac{\pi}{n} + \ \ldots + \frac{1}{2} \times 2^{n-1} \times 2^n \times \sin\frac{\pi}{n}$ ***M1A1***

Note: Award ***M1*** for powers of 2, ***A1*** for any correct expression including both the first and last term.

$= \sin\frac{\pi}{n} \times \left(2^0 + 2^2 + 2^4 + \ \ldots + 2^{n-2}\right)$

identifying a geometric series with common ratio $2^2 (= 4)$ ***(M1)A1***

$= \frac{1-2^{2n}}{1-4} \times \sin\frac{\pi}{n}$ ***M1***

Note: Award ***M1*** for use of formula for sum of geometric series.

$= \frac{1}{3}(4^n - 1)\sin\frac{\pi}{n}$ ***A1***

[6 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]

Find the values of *n* such that $\left(1 + \sqrt{3}\mathrm{i}\right)^n$ is a real number.

Markscheme

EITHER

changing to modulus-argument form

$r = 2$

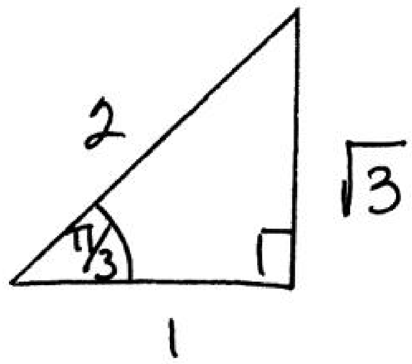
$\theta = \arctan\sqrt{3} = \frac{\pi}{3}$ ***(M1)A1***

$\Rightarrow 1 + \sqrt{3}^n = 2^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$ ***M1***

if $\sin\frac{n\pi}{3} = 0 \Rightarrow n = \{0,\ \pm 3,\ \pm 6,\ \ldots\}$ ***(M1)A1*** ***N2***

OR

$\theta = \arctan\sqrt{3} = \frac{\pi}{3}$ ***(M1)(A1)***



M1

$n \in \mathbb{R} \Rightarrow \frac{n\pi}{3} = k\pi,\ k \in \mathbb{Z}$ ***M1***

$\Rightarrow n = 3k,\ k \in \mathbb{Z}$ ***A1*** ***N2***

[5 marks]

Examiners report

Some candidates did not consider changing the number to modulus-argument form. Among those that did this successfully, many considered individual values of *n*, or only positive values. Very few candidates considered negative multiples of 3.

Part A. (a). Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$. [12]

(b) Draw these roots on an Argand diagram.

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

Part B. (a). Expand and simplify $(x-1)(x^4+x^3+x^2+x+1)$. [13]

(b) Given that b is a root of the equation $z^5 - 1 = 0$ which does not lie on the real axis in the Argand diagram, show that $1 + b + b^2 + b^3 + b^4 = 0$.

(c) If $u = b + b^4$ and $v = b^2 + b^3$ show that

(i) $u + v = uv = -1$;

(ii) $u - v = \sqrt{5}$, given that $u - v > 0$.

Markscheme

Part A. (a). $z = (1 - \mathrm{i})^{\frac{1}{4}}$

Let $1 - \mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\Rightarrow r = \sqrt{2}$ ***A1***

$\theta = -\frac{\pi}{4}$ ***A1***

$z = \left(\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right)^{\frac{1}{4}}$ ***M1***

$= \left(\sqrt{2}\left(\cos\left(-\frac{\pi}{4} + 2n\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{4} + 2n\pi\right)\right)\right)^{\frac{1}{4}}$

$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{\pi}{16} + \frac{n\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{16} + \frac{n\pi}{2}\right)\right)$ ***M1***

$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{\pi}{16}\right)\right)$

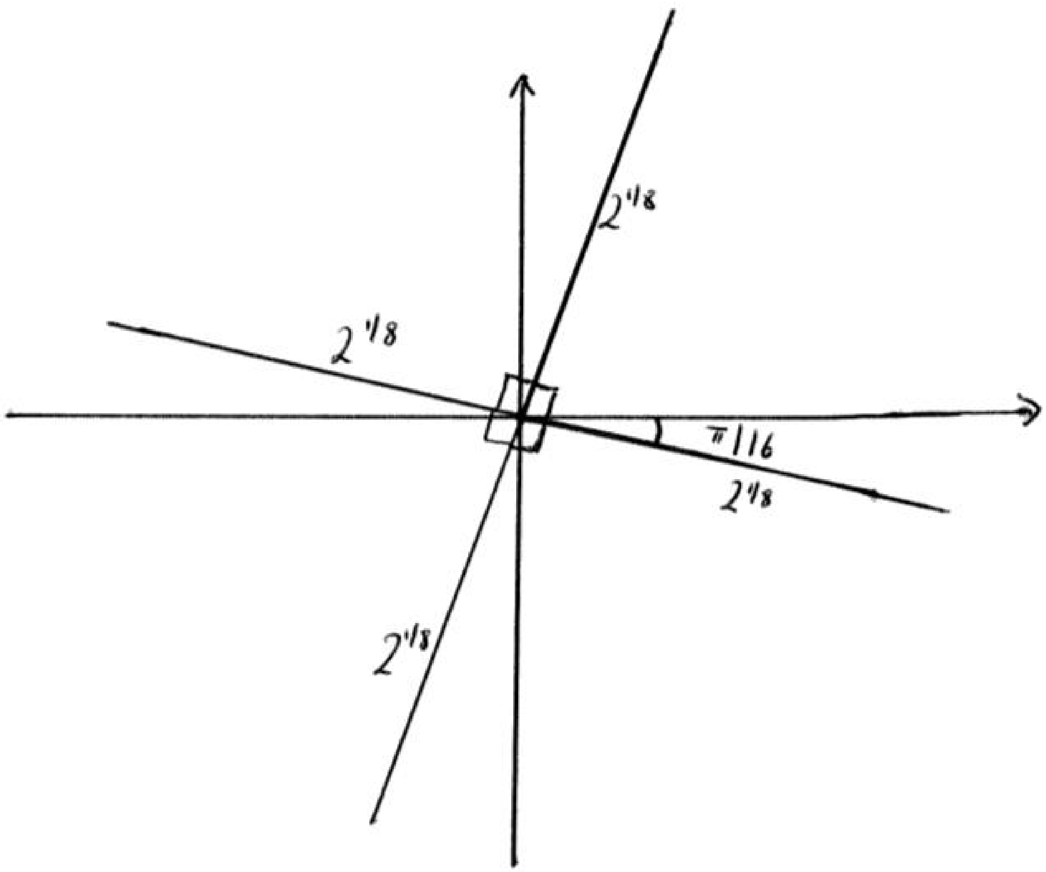
Note: Award ***M1*** above for this line if the candidate has forgotten to add 2π and no other solution given.

$= 2^{\frac{1}{8}}\left(\cos\left(\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)$
$= 2^{\frac{1}{8}}\left(\cos\left(\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)$
$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{9\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{16}\right)\right)$ ***A2***

Note: Award ***A1*** for 2 correct answers. Accept any equivalent form.

[6 marks]

(b)



A2

Note: Award ***A1*** for roots being shown equidistant from the origin and one in each quadrant.
A1 for correct angular positions. It is not necessary to see written evidence of angle, but must agree with the diagram.

[2 marks]

(c) $\frac{z_2}{z_1} = \frac{2^{\frac{1}{8}}\left(\left(\cos\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)}{2^{\frac{1}{8}}\left(\left(\cos\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)}$ ***M1A1***

$= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$ ***(A1)***

$= \mathrm{i}$ ***A1*** ***N2***

$(\Rightarrow a = 0,\ b = 1)$

[4 marks]

Part B. (a) $(x-1)(x^4 + x^3 + x^2 + x + 1)$

$= x^5 + x^4 + x^3 + x^2 + x - x^4 - x^3 - x^2 - x - 1$ ***(M1)***

$= x^5 - 1$ ***A1***

[2 marks]

(b) b is a root

$f(b) = 0$

$b^5 = 1$ ***M1***

$b^5 - 1 = 0$ ***A1***

$(b-1)(b^4 + b^3 + b^2 + b + 1) = 0$

$b \neq 1$ ***R1***

$1 + b + b^2 + b^3 + b^4 = 0$ as shown. ***AG***

[3 marks]

(c) (i) $u + v = b^4 + b^3 + b^2 + b = -1$ ***A1***

$uv = (b + b^4)(b^2 + b^3) = b^3 + b^4 + b^6 + b^7$ ***A1***

Now $b^5 = 1$ ***(A1)***

Hence $uv = b^3 + b^4 + b + b^2 = -1$ ***A1***

Hence $u + v = uv = -1$ ***AG***

(ii) $(u - v)^2 = (u^2 + v^2) - 2uv$ ***(M1)***

$= \left((u+v)^2 - 2uv\right) - 2uv \quad \left(= (u+v)^2 - 4uv\right)$ ***(M1)A1***

Given $u - v > 0$

$u - v = \sqrt{(u+v)^2 - 4uv}$

$= \sqrt{(-1)^2 - 4(-1)}$

$= \sqrt{1+4}$ ***A1***

$= \sqrt{5}$ ***AG***

Note: Award ***A0*** unless an indicator is given that $u - v = -\sqrt{5}$ is invalid.

[8 marks]

Total [13 marks]

Examiners report

Part A. The response to Part A was disappointing. Many candidates did not know that they had to apply de Moivre's theorem and did not appreciate that they needed to find four roots.

Part B.

Part B started well for most candidates, but in part (b) many candidates did not appreciate the significance of b not lying on the real axis. A majority of candidates started (c) (i) and many fully correct answers were seen. Part (c) (ii) proved unsuccessful for all but the very best candidates.

Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram.

(c) (i) Expand and simplify $F(z) = (z-1)(z-\omega)(z-\omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

Markscheme

(a) (i) $\omega^3 = \left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)^3$

$= \cos\left(x \times \frac{2\pi}{3}\right) + \mathrm{i}\sin\left(3 \times \frac{2\pi}{3}\right)$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***A1***

$= 1$ ***AG***

(ii) $1 + \omega + \omega^2 = 1 + \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)$ ***M1A1***

$= 1 + -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} - \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$ ***A1***

$= 0$ ***AG***

[5 marks]

(b) (i) $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)}$

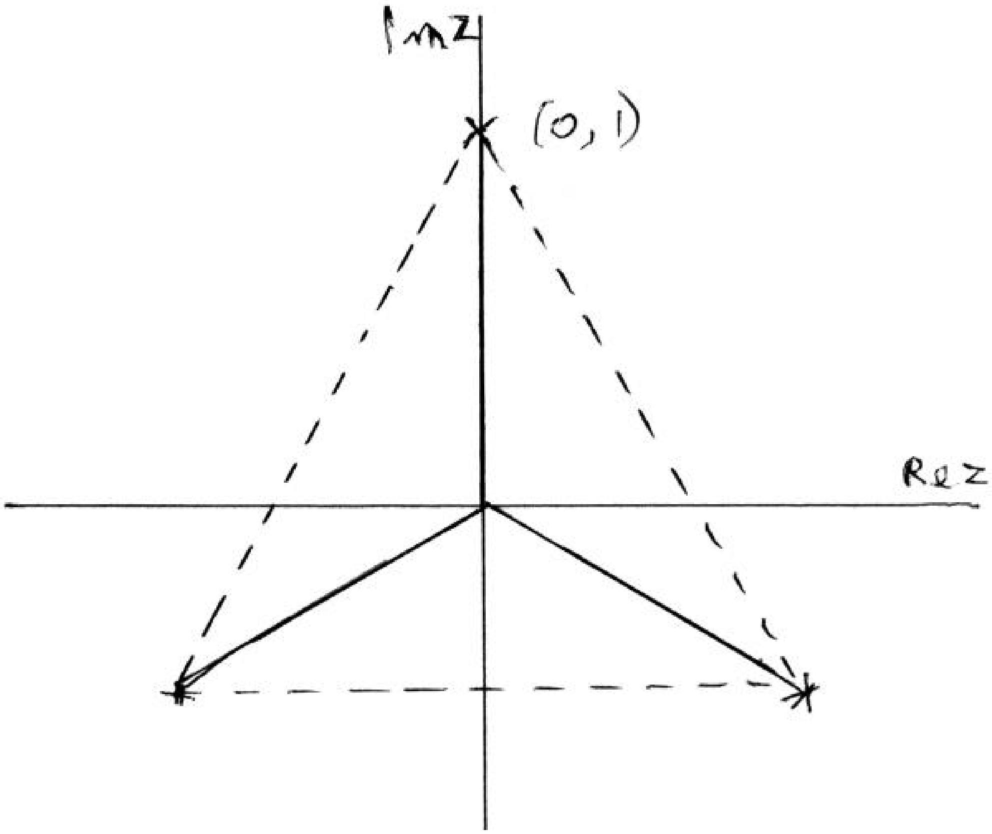
$= \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ ***(M1)***

$= \left(\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}\right)\right)$

$= \mathrm{e}^{\mathrm{i}\theta}(1 + \omega + \omega^2)$ ***A1***

$= 0$ ***AG***

(ii)



A1A1

Note: Award ***A1*** for one point on the imaginary axis and another point marked with approximately correct modulus and argument. Award ***A1*** for third point marked to form an equilateral triangle centred on the origin.

[4 marks]

(c) (i) attempt at the expansion of at least two linear factors ***(M1)***

$(z-1)z^2 - z(\omega+\omega^2) + \omega^3$ or equivalent ***(A1)***

use of earlier result ***(M1)***

$F(z) = (z-1)(z^2+z+1) = z^3-1$ ***A1***

(ii) equation to solve is $z^3 = 8$ ***(M1)***

$z = 2,\ 2\omega,\ 2\omega^2$ ***A2***

Note: Award ***A1*** for 2 correct solutions.

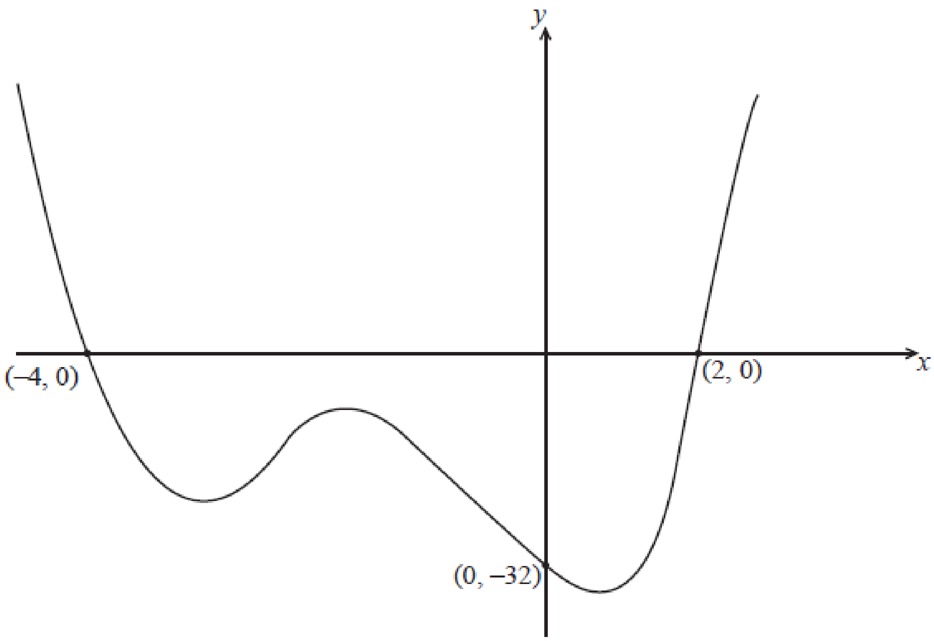
[7 marks]

Total [16 marks]

Examiners report

Most candidates were able to make a meaningful start to part (a) with many fully correct answers seen. Part (b) was the exact opposite with the majority of candidates not knowing what was required and failing to spot the connection to part (a). Candidates made a reasonable start to part (c), but often did not recognise the need to use the result that $1 + \omega + \omega^2 = 0$. This meant that most candidates were unable to make any progress on part (c) (ii).

The graph of a polynomial function *f* of degree 4 is shown below.



A.a Given that $(x + \mathrm{i}y)^2 = -5 + 12\mathrm{i},\ x,\ y \in \mathbb{R}$. Show that [2]

(i) $x^2 - y^2 = -5$;
(ii) $xy = 6$.

A.b Hence find the two square roots of $-5 + 12\mathrm{i}$. [5]

A.c For any complex number z , show that $(z^*)^2 = (z^2)^*$. [3]

A.d Hence write down the two square roots of $-5 - 12\mathrm{i}$. [2]

B.a Explain why, of the four roots of the equation $f(x) = 0$, two are real and two are complex. [2]

B.b The curve passes through the point $(-1,\ -18)$. Find $f(x)$ in the form [5]

$f(x) = (x - a)(x - b)(x^2 + cx + d),$ where $a,\ b,\ c,\ d \in \mathbb{Z}$.

B.c Find the two complex roots of the equation $f(x) = 0$ in Cartesian form. [2]

B.d Draw the four roots on the complex plane (the Argand diagram). [2]

B.e Express each of the four roots of the equation in the form $r\mathrm{e}^{\mathrm{i}\theta}$. [6]

Markscheme

A.a (i) $(x + \mathrm{i}y)^2 = -5 + 12\mathrm{i}$

$x^2 + 2\mathrm{i}xy + \mathrm{i}^2y^2 = -5 + 12\mathrm{i}$ ***A1***

(ii) equating real and imaginary parts ***M1***

$x^2 - y^2 = -5$ ***AG***
$xy = 6$ ***AG***
[2 marks]

A.b. substituting ***M1***

EITHER
$x^2 - \frac{36}{x^2} = -5$
$x^4 + 5x^2 - 36 = 0$ ***A1***
$x^2 = 4,\ -9$ ***A1***
$x = \pm 2$ and $y = \pm 3$ ***(A1)***
OR
$\frac{36}{y^2} - y^2 = -5$
$y^4 - 5y^2 - 36 = 0$ ***A1***
$y^2 = 9,\ -4$ ***A1***
$y^2 = \pm 3$ **and** $x = \pm 2$ ***(A1)***
Note: Accept solution by inspection if completely correct.

THEN
the square roots are $(2 + 3\mathrm{i})$ and $(-2 - 3\mathrm{i})$ ***A1***
[5 marks]

A.c. **EITHER**

consider $z = x + \mathrm{i}y$
$z^* = x - \mathrm{i}y$
$(z^*)^2 = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***
$(z^2) = x^2 - y^2 + 2\mathrm{i}xy$ ***A1***
$(z^2)^* = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***
$(z^*)^2 = (z^2)^*$ ***AG***
OR
$z^* = r\mathrm{e}^{-\mathrm{i}\theta}$
$(z^*)^2 = r^2\mathrm{e}^{-2\mathrm{i}\theta}$ ***A1***
$z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ ***A1***
$(z^2)^* = r^2\mathrm{e}^{-2\mathrm{i}\theta}$ ***A1***
$(z^*)^2 = (z^2)^*$ ***AG***
[3 marks]

A.d. $(2 - 3\mathrm{i})$ and $(-2 + 3\mathrm{i})$ ***A1A1***

[2 marks]

B.a. the graph crosses the *x*-axis twice, indicating two real roots ***R1***

since the quartic equation has four roots and only two are real, the other two roots must be complex ***R1***
[2 marks]

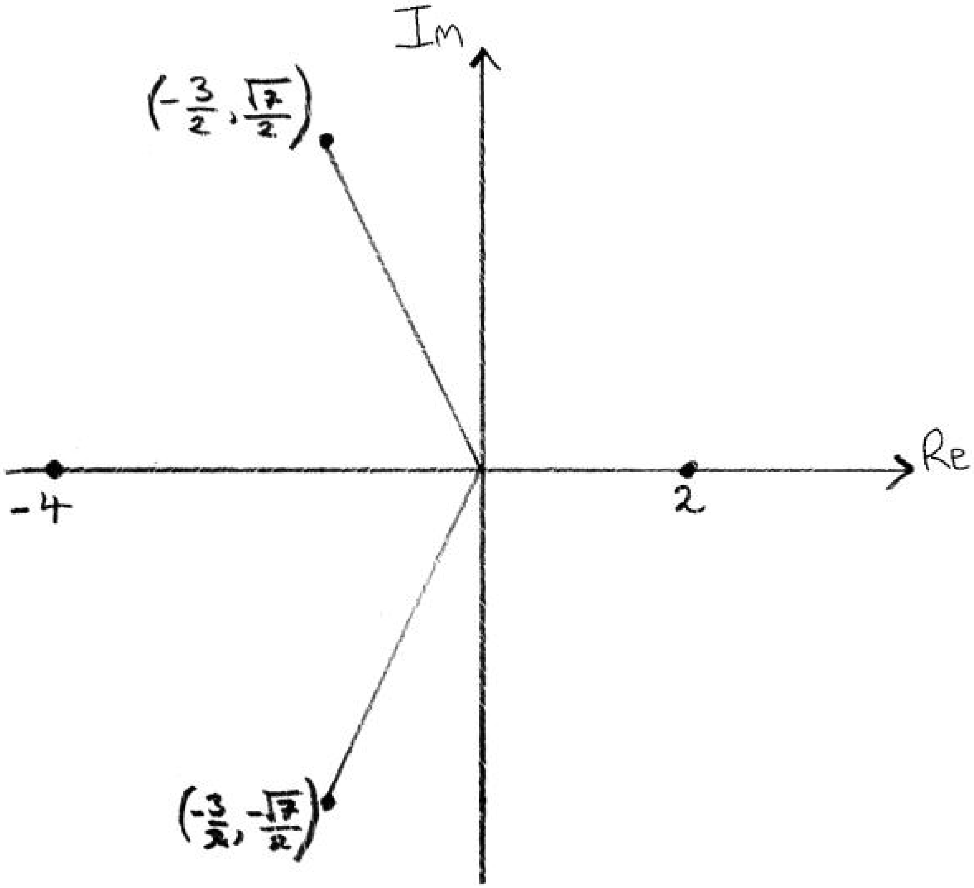
B.b. $f(x) = (x + 4)(x - 2)(x^2 + cx + d)$ ***A1A1***

$f(0) = -32 \Rightarrow d = 4$ ***A1***
Since the curve passes through $(-1,\ -18)$,
$-18 = 3 \times (-3)(5 - c)$ ***M1***
$c = 3$ ***A1***
Hence $f(x) = (x + 4)(x - 2)(x^2 + 3x + 4)$
[5 marks]

B.c. $x = \frac{-3 \pm \sqrt{9 - 16}}{2}$ ***(M1)***

$\Rightarrow x = -\frac{3}{2} \pm \mathrm{i}\frac{\sqrt{7}}{2}$ ***A1***
[2 marks]

B.d.



A1A1

Note: Accept points or vectors on complex plane.
Award ***A1*** for two real roots and ***A1*** for two complex roots.

[2 marks]

B.e. real roots are $4\mathrm{e}^{\mathrm{i}\pi}$ and $2\mathrm{e}^{\mathrm{i}0}$ ***A1A1***

considering $-\frac{3}{2} \pm \mathrm{i}\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$ ***A1***

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$ ***M1***

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$ ***A1***

$\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$ ***A1***

Note: Accept arguments in the range $-\pi$ to π or 0 to 2π .
Accept answers in degrees.

[6 marks]

Examiners report

A.a. Since (a) was a 'show that' question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained.

Many candidates just wrote

$$(x + \mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy = -5 + 12\mathrm{i}$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre’s Theorem to find the square roots were given no credit since the question stated ‘hence’.

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B.a In (a), the explanations were often unconvincing. Candidates were expected to make it clear that the two intersections with the *x*-axis gave two real roots and, since the polynomial was a quartic and therefore had four zeros, the other two roots must be complex. Candidates who made vague statements such as ‘the graph shows two real roots’ were not given full credit. In (b), most candidates stated the values of *a* and *b* correctly but algebraic errors often led to incorrect values for the other parameters. Candidates who failed to solve (b) correctly were unable to solve (c), (d) and (e) correctly although follow through was used where possible.

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